Mathematical Statistics

Section 2

Preface:

I thank Allah Almighty who gave me a chance to serve the students in the field of Mathematics. I also thank the Holy Prophet Hazrat Muhammad (PBUH), who is forever a torch of guidance for humanity, without his goodwill nothing can be achieved. I am very grateful to my friends who encouraged me during the preparation of these notes. Further, I am thankful to my respected teacher **Prof. Dr. Hafiz Muhammad Khalid** for his kind support in this regard.

I feel myself lucky enough that my notes are popular all over Pakistan. Suggestions, comments, and indications of errors for the improvement will be highly appreciated.

I pray that my efforts will help the students in achieving their future objectives.

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Central Limit Theorem.

Statement.

"If a variable X from a population has mean μ and finite variance σ^2, then the sampling distribution of the sample mean $\bar{X}$ approaches a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$ as the sample size n approaches infinity."

OR:

"If a random variable defined on an infinite population has mean μ and variance σ^2 and a sample of n independent observations $X_1, X_2, \cdots, X_n$ is selected from this population, then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ follows a standard normal distribution as 'n' approaches infinity."

Proof.

The moment generating function about mean for a r.v. X is given by

$$M_x(t) = E\left(e^{(X-\mu)t}\right)$$

$$= 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \frac{\mu_3 t^3}{3!} + \mu_4 \frac{t^4}{4!} + \cdots$$

But $\mu_1 = 0$ and $\mu_2 = \sigma^2$

Then, $$M_x(t) = 1 + \sigma^2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \cdots \quad \text{——— (1)}$$

Let us define a r.v. Y as

$$Y = \frac{X-\mu}{\sigma\sqrt{n}} \quad \text{——— (A)}$$

Then the m.g.f. of Y is

$$M_y(t) = E\left(e^{(X-\mu)t/\sigma\sqrt{n}}\right)$$

$$= M_x(t/\sigma\sqrt{n}) \quad \text{——— (2)}$$

Replacing t by $t/\sigma\sqrt{n}$ in eq (1), we get

$$M_x(t/\sigma\sqrt{n}) = 1 + \frac{\sigma^2}{2!}\left(t/\sigma\sqrt{n}\right)^2 + \frac{\mu_3}{3!}\left(t/\sigma\sqrt{n}\right)^3 + \frac{\mu_4}{4!}\left(t/\sigma\sqrt{n}\right)^4 + \dots$$

$$= 1 + \frac{\cancel{\sigma^2}}{2!}\cdot\frac{t^2}{\cancel{\sigma^2}n} + \frac{\mu_3}{3!}\frac{t^3}{\sigma^3 n\sqrt{n}} + \frac{\mu_4}{4!}\frac{t^4}{\sigma^4 n^2} + \dots$$

$$= 1 + \frac{t^2}{2n} + \frac{\mu_3 t^3}{3!\sigma^3 n\sqrt{n}} + \frac{\mu_4 t^4}{4!\sigma^4 n^2} + \dots$$

Now equation (2) becomes

$$M_y(t) = 1 + \frac{t^2}{2n} + \frac{\mu_3 t^3}{3!\sigma^3 n\sqrt{n}} + \frac{\mu_4}{4!\sigma^4 n^2} + \dots \quad \text{——— (3)}$$

Let us define another variable Z as a linear function of $\bar{X}$ as

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

As $\bar{X}-\mu = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)$

So $Z = \frac{1}{n}\sum_{i=1}^{n}\frac{X_i-\mu}{\sigma/\sqrt{n}}$

$$\therefore = \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma\sqrt{n}} = \sum_{i=1}^{n} Y_i \qquad \text{using (A)}$$

$$\Rightarrow Z = \sum_{i=1}^{n} Y_i$$

Then the m.g.f. of Z is

$$M_Z(t) = E(e^{tZ})$$
$$= E(e^{t\sum_{i=1}^{n} Y_i})$$
$$= E(e^{tY_1 + tY_2 + \cdots + tY_n})$$
$$= E(e^{tY_1} \cdot e^{tY_2} \cdots e^{tY_n})$$
$$= E(e^{tY_1}) \cdot E(e^{tY_2}) \cdots E(e^{tY_n})$$
$$= \prod_{i=1}^{n} E(e^{tY_i})$$
$$= \prod_{i=1}^{n} M_{Y_i}(t)$$
$$= \prod_{i=1}^{n} M_Y(t) \qquad \because Y_i\text{'s are independent and identically distributed.}$$

$$\Rightarrow M_Z(t) = (M_Y(t))^n \qquad \text{--- (4)}$$

Subsituting (3) and (4), we get

$$M_Z(t) = \left(1 + \frac{t^2}{2n} + \frac{\mu_3 t^3}{3!\,\sigma^3 n\sqrt{n}} + \frac{\mu_4 t^4}{4!\,\sigma^4 n^2} + \cdots\right)^n$$

Taking ln on both sides, we get

$$\ln M_Z(t) = n \ln\left(1 + \frac{t^2}{2n} + \frac{\mu_3 t^3}{3!\,\sigma^3 n\sqrt{n}} + \cdots\right)$$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Therefore,

$$\ln M_z(t) = n\left[\left(\frac{t^2}{2n} + \frac{t^3 \mu_3}{3!\,\sigma^3 n\sqrt{n}} + \cdots\right) - \frac{1}{2}\left(\frac{t^2}{2n} + \frac{t^3}{3!\,\sigma^3 n\sqrt{n}} + \cdots\right)^2 + \frac{1}{3}\left(\frac{t^2}{2n} + \frac{t^3}{3!\,\sigma^3 n\sqrt{n}} + \cdots\right) - \cdots\right]$$

$$= n\left[\frac{t^2}{2n} + \frac{\mu_3 t^3}{3!\,\sigma^3 n\sqrt{n}} + \cdots\right]$$

$$= \frac{t^2}{2} + \frac{\mu_3 t^3}{3!\,\sigma^3\sqrt{n}} + \text{higher powers of } \frac{1}{n}$$

Taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} \ln M_z(t) = \lim_{n\to\infty}\left(\frac{t^2}{2} + \frac{\mu_3 t^3}{3!\,\sigma^3\sqrt{n}} + \text{h.p of } 1/n\right)$$

$$= \frac{t^2}{2} + 0 + \cdots$$

$$\Rightarrow \lim_{n\to\infty} \ln M_z(t) = t^2/2$$

$$\Rightarrow \lim_{n\to\infty} M_z(t) = e^{t^2/2}$$

which is moment generating function of standard normal distribution.

Hence, $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ follows a standard normal distribution as n approaches infinity.

The Chi-Square (χ^2) Distribution.

Let $Z_1, Z_2, \ldots, Z_n$ be normally and independently distributed variables with zero means and unit variances. Then a random variable expressed by the quantity

$$\chi^2 = \sum_{i=1}^{n} Z_i^2$$

is defined as a chi-square random variable with n-degrees of freedom.

That is, a χ^2 random variable is defind as the sum of squares of n idependent standard normal random variables.

It density function has the following form:

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma(n/2)} (\chi^2)^{\frac{n}{2}-1} e^{-\chi^2/2} \; ; 0 < \chi^2 < \infty$$

Random variables having the above density function are said to possess the chi-square distribution with n degrees of freedom.

It is denoted by $\chi^2_{(n)}$, where the parameter n called the degree of freedom.

Derivation.

To derive the distribution of χ^2, we use the m.g.f. technique.

The m.g.f. of chi-square is given by

A/2013
A/2018 Obtain m.g.f of χ^2-distribution and hence find 1st 3 moments about mean.

$$M_0(t) = E(e^{t\chi^2})$$

$$= E(e^{t\sum_{i=1}^{n} z_i^2})$$

$$= E(e^{tZ_1^2 + tZ_2^2 + \dots + tZ_n^2})$$

$$= E(e^{tZ_1^2} . e^{tZ_2} \dots e^{tZ_n^2})$$

$$= E(\prod_{i=1}^{n} e^{tZ_i^2})$$

$$= \prod_{i=1}^{n} E(e^{tZ_i^2})$$ as Z_i's are all independent.

$$\Rightarrow M_0(t) = \prod_{i=1}^{n} E(e^{tZ_i^2}) \quad \text{———} \quad (1)$$

Now, $$E(e^{tZ_i^2}) = \int_{-\infty}^{\infty} e^{tz^2} . \phi(z)\, dz$$

$$= \int_{-\infty}^{\infty} e^{tz^2} . \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2(t-\frac{1}{2}) z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1-2t) z^2/2}\, dz \quad \text{even}$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} e^{-(1-2t) z^2/2}\, dz$$

Put $(1-2t) z^2/2 = y$

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$$(1-2t)z^2 = 2y$$

$$\Rightarrow \quad z^2 = \frac{2y}{1-2t} \qquad \Rightarrow z = \sqrt{\frac{2y}{1-2t}} = \frac{\sqrt{2y}}{\sqrt{1-2t}}$$

$$\Rightarrow dz = \frac{1}{\sqrt{1-2t}} \cdot \frac{1}{\cancel{2}} (2y)^{-1/2} \cancel{2}\, dy$$

$$\Rightarrow \quad dz = \frac{1}{\sqrt{1-2t}} \cdot \frac{1}{\sqrt{2y}} \, dy$$

when $z \longrightarrow 0, \quad y \longrightarrow 0$

$\qquad z \longrightarrow \infty, \quad y \longrightarrow \infty$

Then $$E(e^{tZ_i^2}) = \frac{\cancel{2}}{\sqrt{\cancel{2}\pi}} \int_0^\infty e^{-y} \frac{dy}{\sqrt{1-2t}\sqrt{\cancel{2}y}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y}\, y^{-1/2}\, dy}{\sqrt{1-2t}}$$

$$= \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_0^\infty e^{-y}\, y^{\frac{1}{2}-1}\, dy$$

$$= \frac{1}{\sqrt{\pi}\sqrt{1-2t}}\, \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{\cancel{\sqrt{\pi}}\sqrt{1-2t}}\, \cancel{\sqrt{\pi}}$$

$$\Rightarrow E(e^{tZ_i^2}) = \frac{1}{\sqrt{1-2t}} \quad \text{for } t < 1/2$$

Thus equation ① becomes

$$M_o(t) = \prod_{i=1}^{n} \frac{1}{\sqrt{1-2t}} = \left(\frac{1}{\sqrt{1-2t}}\right)^n$$

$$\Rightarrow \boxed{M_o(t) = (1-2t)^{-n/2}} \quad \text{for } t < \frac{1}{2}.$$

The m.g.f. of Gamma distribution

$$f(x) = \frac{a^m}{\Gamma(m)} x^{m-1} e^{-ax} \quad ; 0 \le x < \infty$$

is $M_o(t) = (1 - t/a)^{-m}$ for $t < 1$

Comparing these m.g.f.s, we get

$$m = \frac{n}{2} \quad \text{and} \quad \frac{1}{a} = 2 \quad \text{or} \quad a = \frac{1}{2}$$

Thus, the p.d.f. of χ^2 is

Putting in Gamma di

$$f(\chi^2) = \frac{(1/2)^{n/2}}{\Gamma(n/2)} (\chi^2)^{n/2 - 1} e^{-\chi^2/2} \quad ; 0 < \chi^2 < \infty$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} (\chi^2)^{n/2-1} e^{-\chi^2/2} \quad ; 0 < \chi^2 < \infty$$

Note:
Verification of p.d.f. of χ^2-distribution is given above.

Measure of symmetry:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

Measure of Kurtosis:

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

Properties:

1. The chi-square is a continuous distribution ranging from 0 to $+\infty$.
2. The mean of chi-square distribution is equal to the number of degrees of freedom and its variance is equal to twice the number of degrees of freedom.

Mean: A/2009, A/2011,

The moment generating function is given by

$$M_0(t) = E(e^{tx^2}) \quad \text{already calculated.}$$
$$= (1-2t)^{-n/2}, \text{ for } t < \tfrac{1}{2}.$$

اگر $X \sim \chi^2$ ہو تو mean اور variance کو direct لکھ سکتے ہیں۔

As

$$\mu_r' = \frac{d^r}{dt^r} M_0(t)\Big|_{t=0}$$

So

$$\mu_1' = \frac{d^1}{dt^1}(1-2t)^{-n/2}\Big|_{t=0}$$
$$\Rightarrow \mu_1' = \frac{d}{dt}(1-2t)^{-n/2}\Big|_{t=0}$$
$$= -\frac{n}{2}(1-2t)^{-\frac{n}{2}-1}(-2)\Big|_{t=0}$$
$$= n(1-2t)^{-\frac{n}{2}-1}\Big|_{t=0}$$
$$\Rightarrow \boxed{\mu_1' = n = \text{mean}}$$

$$\mu_2' = \frac{d^2}{dt^2}(1-2t)^{-n/2}\Big|_{t=0}$$

$$= \frac{d}{dt} n(1-2t)^{-\frac{n}{2}-1} \Big|_{t=0}$$

$$= n\left(-\frac{n}{2}-1\right)(1-2t)^{-\frac{n}{2}-2}(-2) \Big|_{t=0}$$

$$= 2n\left(\frac{n}{2}+1\right)(1-2t)^{-n/2-2} \Big|_{t=0}$$

$$\Rightarrow \mu_2' = 2n\left(\frac{n}{2}+1\right)$$

$$= \cancel{2}n\left(\frac{n+2}{\cancel{2}}\right) = n(n+2)$$

$$\because \mu_2 = \mu_2' - (\mu_1')^2$$

$$= n(n+2) - n^2$$

$$= \cancel{n^2} + 2n - \cancel{n^2}$$

$$\Rightarrow \boxed{\mu_2 = 2n = \text{Variance}}$$

A/2011

3 The moments about origin of χ^2-dis are given by

The moment g.f. is

$$M_0(t) = (1-2t)^{-n/2} \quad \text{for } t < 1/2.$$

$$\mu_r' = \frac{d^r}{dt^r} M_0(t) \Big|_{t=0}$$

$$= \frac{d^r}{dt^r} (1-2t)^{-n/2} \Big|_{t=0}$$

$$\Rightarrow \mu_1' = n$$ already

$$\mu_2' = 2n$$ calculated.

$$\mu_3' = \frac{d}{dt}\left(n(n+2)(1-2t)^{-\frac{n}{2}-2}\right)\Big|_{t=0}$$

$$= n(n+2)\left(-\frac{n}{2}-2\right)(1-2t)^{-\frac{n}{2}-3}(-2)\Big|_{t=0}$$

$$= 2n(n+2)\left(\frac{n}{2}+2\right)(1-2t)^{-n/2-3}\Big|_{t=0}$$

$$= n(n+2)(n+4)(1)$$

$$\Rightarrow \mu_3' = n(n+2)(n+4)$$

and

$$\mu_4' = \frac{d}{dt}\, n(n+2)(n+4)(1-2t)^{-\frac{n}{2}-3}\Big|_{t=0}$$

$$= n(n+2)(n+4)\left(-\frac{n}{2}-3\right)(1-2t)^{-n/2-4}(-2)\Big|_{t=0}$$

$$= n(n+2)(n+4)(n+6)$$

OR

$$M_0(t) = (1-2t)^{-\frac{n}{2}} \quad \text{for } t < 1/2.$$

Expanding $M_0(t)$ in power series, we get

$$M_0(t) = 1 + \left(\frac{-n}{2}\right)(-2t) + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)}{2!}(-2t)^2$$

$$+ \frac{(-n/2)(-n/2-1)(-n/2-2)}{3!}(-2t)^3 + \cdots$$

$$+ \frac{(-n/2)(-n/2-1)\cdots(-n/2-(s-1))}{s!}(-2t)^s$$

$$= 1+nt + \frac{(n/2)\left(\frac{n+2}{2}\right)}{2!}4t^2 + \frac{(n/2)\left(\frac{n+2}{2}\right)\left(\frac{n+4}{2}\right)}{3!}(8t^3)$$

$$+ \cdots + \frac{(n/2)\left(\frac{n+2}{2}\right)\cdots\left(\frac{n+s-1}{2}\right)}{s!}(2t)^s$$

$$= 1+nt + n(n+2)\frac{t^2}{2!} + n(n+2)(n+4)\frac{t^3}{3!} + n(n+2)(n+4)(n+6)\frac{t^4}{4!} + \cdots$$

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$+ \; n(n+2)(n+4)(n+6)\ldots(n+s-1)\dfrac{t^s}{s!}$ ——— ①

The general expression for m.g.f is

$M_0(t) = 1 + \mu_1' t + \mu_2' \dfrac{t^2}{2!} + \mu_3' \dfrac{t^3}{3!} + \mu_4' \dfrac{t^4}{4!} + \ldots$

$+ \mu_s' \dfrac{t^s}{s!}$ ——— ②

Comparing like power of t in ① and ②, we get

$\mu_1' = n \quad ; \quad \mu_2' = n(n+2)$

$\mu_3' = n(n+2)(n+4)$

$\mu_4' = n(n+2)(n+4)(n+6)$

and

$\mu_s' = n(n+2)(n+4)\ldots(n+s-1)$

Also

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= n(n+2)(n+4) - 3n(n+2)n + 2n^3$

$= (n^2+2n)(n+4) - 3n^2(n+2) + 2n^3$

$= \cancel{n^3} + \cancel{4n^2} + \cancel{2n^2} + 8n - \cancel{3n^3} - \cancel{6n^2} + \cancel{2n^3}$

$\boxed{\mu_3 = 8n}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= n(n+2)(n+4)(n+6) - 4n(n+2)(n+4)n$

$+ 6n(n+2)n^2 - 3n^4$

$= n(n^2+6n+8)(n+6) - 4n^2(n^2+6n+8)$

$+ 6n^3(n+2) - 3n^4$

$= (n^2+6n+8)(n^2+6n-4n^2)+6n^4+12n^3-3n^4$

$= (n^2+6n+8)(6n-3n^2)+3n^4+12n^3$

$= \cancel{6n^3}+36n^2+48n-\cancel{3n^4}-\cancel{18n^3}-24n^2+\cancel{3n^4}+\cancel{12n^3}$

$= 12n^2+48n$

5/2018. Discuss the skewness and kurtosis of χ^2-distri.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{64n^2}{8n^3} = 8/n$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{12n^2+48n}{4n^2} = 3+\frac{12}{n}.$$

Cumulant Generating function.

The cumulant g.f is given by

$$K(t) = \ln M_0(t)$$
$$= \ln(1-2t)^{-n/2}$$
$$= -\frac{n}{2}\ln(1-2t)$$

Since $\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots$

Therefore,

$$K(t) = -\frac{n}{2}\left[-2t-\frac{(+2t)^2}{2}-\frac{(2t)^3}{3}-\frac{(2t)^4}{4}-\cdots\right]$$

$$= \frac{n}{2}\left(2t+\frac{4t^2}{2}+\frac{8t^3}{3}+\frac{16t^4}{4}+\cdots\right)$$

$$= n\left(t+t^2+4\frac{t^3}{3}+8\frac{t^4}{4}+\cdots\right)$$

$$= n\left(\frac{t}{1!}+2\frac{t^2}{2!}+8\frac{t^3}{3!}+48\frac{t^4}{4!}+\cdots\right) \quad (1)$$

The general expression for c.g.f is

$K(t) = k_1 \frac{t}{1!} + k_2 \frac{t^2}{2!} + k_3 \frac{t^3}{3!} + k_3 \frac{t^4}{4!} + \cdots$ ——— ②

Comparing ① and ②, we get

$k_1 = n$; $k_2 = 2n$, $k_3 = 8n$, $k_4 = 48n$

④ The χ^2 - distribution tends to normal distribution as the number of degrees of freedom approaches infinity. A/2013 A/2014

Proof. S/2018

The m.g.f of χ^2 - distribution is

$$M_0(t) = (1-2t)^{-n/2}, \text{ for } t < 1/2. \quad ——— ①$$

and mean and variance of the dis. are $\mu = n$ and $\sigma^2 = 2n$.

Let us define Chi-square standard variable

$$Z = \frac{\chi^2 - \mu}{\sigma} = \frac{\chi^2 - n}{\sqrt{2n}}$$

Then its m.g.f is

$$M_Z(t) = E(e^{tZ})$$

$$= E\left(e^{t\left(\frac{\chi^2 - n}{\sqrt{2n}}\right)}\right)$$

$$= E\left(e^{t\chi^2/\sqrt{2n} - tn/\sqrt{2n}}\right)$$

$$= E\left(e^{t\chi^2/\sqrt{2n}} \cdot e^{-t\sqrt{n/2}}\right)$$

$$= e^{-t\sqrt{n/2}} E\left(e^{t\chi^2/\sqrt{2n}}\right)$$

$$= e^{-t\sqrt{\frac{n}{2}}} M_0\left(t/\sqrt{2n}\right) \quad ——— ②$$

Replacing t by $t/\sqrt{2n}$ in ①, we get

$$M_o(t/\sqrt{2n}) = \left(1-2\,t/\sqrt{2n}\right)^{-n/2}$$

Now equation ② becomes

$$M_z(t) = \left(1-\frac{2t}{\sqrt{2n}}\right)^{-n/2} e^{-t\sqrt{\frac{n}{2}}}$$

Taking natural logs, we get

$$\ln M_z(t) = -\frac{n}{2}\ln\left(1-\frac{2t}{\sqrt{2n}}\right) + \ln e^{-t\sqrt{\frac{n}{2}}}$$

$$= -\frac{n}{2}\left[-\frac{2t}{\sqrt{2n}} - \frac{1}{2}\left(\frac{2t}{\sqrt{2n}}\right)^2 - \frac{1}{3}\left(\frac{2t}{\sqrt{2n}}\right)^3 - \cdots\right] + \ln e^{-t\sqrt{n/2}}$$

$$= \frac{n}{2}\left(\frac{2t}{\sqrt{2n}} + \frac{1}{2}\left(\frac{4t^2}{2n}\right) + \frac{1}{3}\left(\frac{8t^3}{2n\sqrt{2n}}\right) + \cdots\right) - t\sqrt{n/2}$$

$$= \cancel{\sqrt{\frac{n}{2}}\,t} + \frac{t^2}{2} + \frac{2t^3}{3\sqrt{2n}} + \text{higher powers of } \frac{1}{n} \cancel{- t\sqrt{n/2}}$$

$$= \frac{t^2}{2} + \text{higher powers of } \frac{1}{n}$$

Taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} \ln M_z(t) = \lim_{n\to\infty}\left(\frac{t^2}{2} + \text{higher powers of } \frac{1}{n}\right)$$

$$= t^2/2$$

$$\Rightarrow \lim_{n\to\infty} M_Z(t) = e^{t^2/2}$$

which is m.g.f. of standard normal variable.

(v) Additive property A/2012

If X and Y are independent χ^2 random variables with n_1 and n_2 degrees of freedom respectively, then the sum $X+Y$ is a χ^2 r.v. with n_1+n_2 degrees of freedom.

Proof.

Suppose X and Y be two independent χ^2 random variables with n_1 and n_2 degrees of freedom respectively.

Then their m.g.f.s are

$$M_X(t) = (1-2t)^{-n_1/2}, \quad t < 1/2$$

and $$M_Y(t) = (1-2t)^{-n_2/2}$$

Let us define a new r.v. W as

$$W = X+Y$$

Then m.g.f. of W is given by

$$M_W(t) = E(e^{tW})$$

$$= E(e^{t(X+Y)})$$

$$= E(e^{tX}\cdot e^{tY})$$

$$= E(e^{tX})\cdot E(e^{tY}) \qquad \because X \text{ and } Y \text{ are independent}$$

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$$= (1-2t)^{-n_1/2} (1-2t)^{-n_2/2}$$

$$= (1-2t)^{-\frac{n_1}{2} - \frac{n_2}{2}}$$

$$= (1-2t)^{-\left(\frac{n_1+n_2}{2}\right)}$$

But this is the m.g.f. for a χ^2 r.v. with n_1+n_2 degrees of freedom.

Hence proved.

A/2016

(vi) Partitioning Property. State and prove

A χ^2 random variable can be partitioned into two or more than two parts which are also χ^2 r. variables and the sum of their degrees of freedom equals the total degrees of freedom.

Proof.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$. Then the quantity $\sum_{i=1}^{n}(X_i-\mu)^2$ may be expressed by the following simple identity:

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)^2$$

$$= \sum_{i=1}^{n}\left[(X_i-\bar{X})+(\bar{X}-\mu)\right]^2$$

$$= \sum_{i=1}^{n}\left[(X_i-\bar{X})^2+(\bar{X}-\mu)^2+2(X_i-\bar{X})(\bar{X}-\mu)\right.$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + (\bar{x} - \mu)^2 \sum_{i=1}^{n} (1) + 2(\bar{x} - \mu) \sum_{i=1}^{n} (x_i - \bar{x})$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + (\bar{x} - \mu)^2 n + 2(\bar{x} - \mu) \left[\sum_{i=1}^{n} x_i - \bar{x} \sum_{i=1}^{n} (1) \right]$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + (\bar{x} - \mu)^2 n + 2(\bar{x} - \mu) \left\{ \sum_{i=1}^{n} x_i - n\bar{x} \right\}$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + (\bar{x} - \mu)^2 n + 2(\bar{x} - \mu)(n\bar{x} - n\bar{x})$$

$$\because \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

Dividing both sides by σ^2, we get

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} + \frac{n(\bar{x} - \mu)^2}{\sigma^2}$$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right)^2 + n \left(\frac{\bar{x} - \mu}{\sigma} \right)^2$$

Note that, the term $\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2$ is distributed as $\chi^2_{(n)}$ being sum of n standard normal r.v.s.

The term $\sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right)^2$ is χ^2-distribution with $(n-1)$ degrees of freedom and its distribution is independent of $\bar{x}$.

Furthermore, the term $n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$ may be written as $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$ which is $\chi^2_{(1)}$ random variable as square of a standard normal variable.

Thus, we have the identity

$$\chi^2_{(n)} = \chi^2_{(n-1)} + \chi^2_{(1)}$$

Mode.

A/2011 Here $f(\chi^2) = \frac{1}{2^{n/2}\Gamma(n/2)} (\chi^2)^{n/2-1} e^{-\chi^2/2}$

Suppose $X \sim \chi^2_{(n)}$

Then

$$f(x) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-x/2}$$

Diff. w.r.t. x, we get

$$f'(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\left\{\left(\frac{n}{2}-1\right)x^{\frac{n}{2}-2} e^{-x/2} - \frac{1}{2}e^{-x/2} x^{n/2-1}\right\}$$

$$= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\left\{x^{\frac{n}{2}-1} e^{-x/2}\left(\left(\frac{n}{2}-1\right)x^{-1} - \frac{1}{2}\right)\right\}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)}\left\{x^{n/2-1} e^{-x/2}\left(\frac{n-2}{2x} - \frac{1}{2}\right)\right\}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}\left(\frac{n-2}{2x} - \frac{1}{2}\right)$$

Put $f'(x) = 0$

Then $\dfrac{1}{2^{n/2}\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-x/2}\left(\dfrac{n-2}{2x} - \dfrac{1}{2}\right) = 0$

$\Rightarrow \dfrac{n-2}{2x} - \dfrac{1}{2} = 0 \qquad \because \dfrac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} \neq 0$

$\Rightarrow \dfrac{n-2}{2x} = \dfrac{1}{2}$

$\Rightarrow \boxed{n-2 = x}$

Diff. again w.r.t. x, we get

$f''(x) = \dfrac{1}{2^{n/2}\Gamma(n/2)} \dfrac{d}{dx}\left[e^{-x/2}\left\{ \left(\frac{n}{2}-1\right)x^{\frac{n}{2}-2} - \frac{1}{2}x^{\frac{n}{2}-1}\right\}\right]$

$= \dfrac{1}{2^{n/2}\Gamma(n/2)}\left[e^{-x/2}\left\{ \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)x^{n/2-3} - \frac{1}{2}\left(\frac{n}{2}-1\right)x^{\frac{n}{2}-2}\right\} - \frac{1}{2}e^{-x/2}\left\{\left(\frac{n}{2}-1\right)x^{n/2-2} - \frac{1}{2}x^{n/2-1}\right\}\right]$

$= \dfrac{1}{2^{n/2}\Gamma(\frac{n}{2})}\left[\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)e^{-x/2}x^{n/2-3} - \left(\frac{n}{2}-1\right)e^{-x/2}x^{\frac{n}{2}-2} + \frac{1}{4}e^{-x/2}x^{\frac{n}{2}-1}\right]$

$= \dfrac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2}x^{n/2-1}\left[\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)x^{-2} - \left(\frac{n}{2}-1\right)x^{-1} + \frac{1}{4}\right]$

$= \dfrac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2}x^{n/2-1}\left[\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\frac{1}{x^2} - \left(\frac{n}{2}-1\right)\frac{1}{x} + \frac{1}{4}\right]$

Subsituting $x = n-2$ in $f''(x)$, we get

$$f''(x)\Big|_{x=n-2} = \frac{1}{2^{n/2}\Gamma(n/2)} (n-2)^{n/2-1} e^{\frac{2-n}{2}}\left[\frac{(\cancel{n-2})(n-4)}{4}\frac{1}{(n-2)^2} - \left(\frac{\cancel{n-2}}{2}\right)\frac{1}{\cancel{n-2}} + \frac{1}{4}\right]$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} (n-2)^{n/2-1} e^{-\frac{n}{2}+1}\left[\frac{n-4}{4(n-2)} - \frac{1}{2} + \frac{1}{4}\right]$$

$$= \text{''} \quad \text{''} \quad \text{''} \left[\frac{n-4-2(n-2)+n-2}{4(n-2)}\right]$$

$$= \text{''} \quad \text{''} \quad \text{''} \left[\frac{\cancel{n}\cancel{-4}\cancel{-2n}\cancel{+4}\cancel{+n}-2}{4(n-2)}\right]$$

$$\Rightarrow f''(x)\Big|_{x=n-2} = -\frac{1}{2^{n/2}\Gamma(n/2)}(n-2)^{n/2-1} e^{-n/2+1}\frac{1}{2(n-2)}$$

$$= -\frac{1}{2^{n/2+1}\Gamma(n/2)}(n-2)^{\frac{n}{2}-2} e^{-\frac{n}{2}+1} < 0$$

Hence $x = n-2$ is mode.

Harmonic Mean.

$$H.M. = \frac{1}{E(1/\chi^2)}$$

Suppose $X \sim \chi^2_{(n)}$.

Then

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$$

$$E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx$$

$$= \int_0^\infty \frac{1}{x} \cdot \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} x^{n/2-1} e^{-x/2} dx.$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{n/2-2} e^{-x/2} dx.$$

Put $\frac{x}{2} = u \Rightarrow x = 2u \Rightarrow dx = 2du.$

$$= \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)} (2u)^{\frac{n}{2}-2} e^{-u} 2du$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty 2^{n/2-2} u^{n/2-2} e^{-u}\, 2\, du$$

$$= \frac{1}{\cancel{2^{n/2}}\Gamma(n/2)} \cancel{2^{n/2}} \cdot 2^{-2} \cdot 2 \int_0^\infty u^{n/2-2} e^{-u} du$$

$$= \frac{1}{2\Gamma(n/2)} \int_0^\infty e^{-u} u^{(n/2-1)-1} du$$

$$= \frac{1}{2\Gamma(n/2)} \Gamma\left(\frac{n}{2}-1\right)$$

$$= \frac{1}{2(\frac{n}{2}-1)\cancel{\Gamma(\frac{n}{2}-1)}} \cdot \cancel{\Gamma\left(\frac{n}{2}-1\right)} = \frac{1}{n-2}$$

Thus

$H.M = n-2$

The Student's t - Distribution.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal population with mean μ and variance σ^2 and let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, which is the unbiased estimate of σ^2. Then the sampling distribution of the statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

is called Student's t-distribution with $(n-1)$ degrees of freedom.

The statistic t may be written as

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{\bar{x} - \mu}{s/\sqrt{n}} \cdot \frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}}$$

$$= \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma/\cancel{\sqrt{n}}}{s/\cancel{\sqrt{n}}}$$

$$= \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{s}$$

$$= \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

$$= \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \cdot \frac{1}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2}}}$$

$$= \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \cdot \frac{1}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}}$$

$$= Z \cdot \frac{1}{\sqrt{\frac{1}{n-1}\cdot U}}$$ where $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is s.n. variable and $U = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2$

is a χ^2-random variable with $(n-1)$ degrees of freedom.

$$\Rightarrow \quad t = \frac{Z}{\sqrt{\frac{U}{n-1}}}$$

Thus, the t-statistic is the quotient of standard normal variable and the square root of a chi-square random variable divided by its degree of freedom.

Derivation. A/2014, A/2017, S/2018

Let Z be a standard normal r.v. and U, a χ^2 random variable with n degrees of freedom. If Z and U are independent, their joint density will be

$$f(z,u) = g(z).h(u)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \cdot \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} u^{n/2 - 1} e^{-u/2}$$

for $-\infty < z < \infty$ and $0 < u < \infty$

Put $t = z/\sqrt{\frac{u}{n}}$ and $w = u$, so that

$z = t\sqrt{w/n}$ and $w = u$.

Then, by change of variable technique, we have

$$f_{x,y}(x,y) = |J|\, f_{u,n}(x,y)$$

where J is the Jacobian of the transformation.

Thus,

$$J = \frac{\partial(z,u)}{\partial(t,w)} = \begin{vmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{w/n} & t/2\sqrt{wn} \\ 0 & 1 \end{vmatrix}$$

$$\because \frac{\partial z}{\partial w} = \frac{t}{2}\left(\frac{w}{n}\right)^{-1/2}\left(\frac{1}{n}\right)$$

$$= \frac{t}{2n}\frac{1}{\sqrt{w/n}}$$

$$= \frac{t}{2n}\sqrt{n/w}$$

$$= \frac{t}{2}\frac{1}{\sqrt{nw}}$$

$$= \sqrt{w/n}$$

Subsituting these values, we obtain the joint density of t and w as

$$f(t,w) = \sqrt{\frac{w}{n}} \cdot \frac{1}{\sqrt{2\pi}\, 2^{n/2}\Gamma(n/2)} w^{\frac{n}{2}-1} e^{-t^2/2 \, w/n} e^{-w/2}$$

$$= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})\sqrt{2\pi}} \sqrt{\frac{w}{n}}\, w^{n/2-1} e^{-\frac{w}{2}(\frac{t^2}{n}+1)} \quad ; \quad \begin{matrix} -\infty < t < \infty \\ 0 < w < \infty \end{matrix}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)\sqrt{2\pi}} \frac{w^{1/2}}{n^{1/2}} w^{n/2-1} e^{-\frac{w}{2}(\frac{t^2}{n}+1)}$$

$$= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})\sqrt{2\pi n}} w^{\frac{n}{2}-\frac{1}{2}} e^{-\frac{w}{2}(\frac{t^2}{n}+1)}$$

Integrating w.r.t. w, we find the density of t as

$$f_n(t) = \int_0^\infty f(t,w)\, dw$$

$$= \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)\sqrt{2\pi n}} w^{(n-1)/2} e^{-\frac{w}{2}(\frac{t^2}{n}+1)} dw$$

Put

$$\frac{w}{2}(t^2/n+1) = y \Rightarrow w = \frac{1}{(t^2/n+1)} 2y \qquad \text{when } \begin{matrix} w \to 0 \\ y \to 0 \end{matrix}$$

$$\Rightarrow \frac{1}{2}(\frac{t^2}{n}+1)\, dw = dy \qquad \begin{matrix} w \to \infty \\ y \to \infty \end{matrix}$$

Then

$$f_n(t) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})\sqrt{2\pi n}} \int_0^\infty \left(\frac{2y}{1+t^2/n}\right)^{(n-1)/2} e^{-y} \frac{2dy}{1+t^2/n}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)\sqrt{2\pi n}} \int_0^\infty \frac{2^{\frac{n-1}{2}} y^{\frac{n-1}{2}}}{(1+t^2/n)^{(n-1)/2}} e^{-y} \frac{2dy}{1+t^2/n}$$

$$= \frac{1}{2^{n/2}\,\Gamma(n/2)\,\sqrt{2\pi n}} \int_0^\infty \frac{2^{\frac{n}{2}+\frac{1}{2}}\, y^{\frac{n-1}{2}}\, e^{-y}}{(1+t^2/n)^{\frac{n}{2}+\frac{1}{2}}}\, dy$$

$$= \frac{\cancel{2^{n/2}}\,\cancel{2^{1/2}}}{\cancel{2^{n/2}}\,\Gamma(n/2)\,\cancel{2^{1/2}}\sqrt{\pi n}} \int_0^\infty (1+t^2/n)^{-\left(\frac{n+1}{2}\right)} e^{-y}\, y^{\frac{n}{2}-\frac{1}{2}}\, dy$$

$$= \frac{1}{\Gamma(\frac{n}{2})\cdot\sqrt{n\pi}} (1+t^2/n)^{-\left(\frac{n+1}{2}\right)} \int_0^\infty e^{-y}\, y^{\left(\frac{n}{2}+\frac{1}{2}\right)-1}\, dy$$

$$= \frac{1}{\Gamma(n/2)\,\sqrt{n\pi}} (1+t^2/n)^{-\left(\frac{n+1}{2}\right)}\, \Gamma\left(\frac{n}{2}+\frac{1}{2}\right)$$

$$= \frac{1}{\cancel{\Gamma(n/2)}\sqrt{n\pi}} (1+t^2/n)^{-\left(\frac{n+1}{2}\right)}\, \frac{\cancel{\Gamma(n/2)}\cdot\Gamma(1/2)}{\beta(\frac{n}{2},\frac{1}{2})}$$

$$= \frac{1}{\beta(\frac{n}{2},\frac{1}{2})}\, \frac{(1+t^2/n)^{-\left(\frac{n+1}{2}\right)}}{\sqrt{n\cancel{\pi}}}\, \cancel{\sqrt{\pi}}$$

$$= \frac{1}{\beta(\frac{n}{2},\frac{1}{2})}\, \frac{(1+t^2/n)^{-\left(\frac{n+1}{2}\right)}}{\sqrt{n}}$$

$$\Rightarrow f_n(t) = \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} (1+t^2/n)^{-(n+1)/2} \quad \text{for } -\infty < t < \infty$$

which is called Student's t-distribution with n degrees of freedom.

Properties.

① The t- distribution is continuous and symmetric about the value $t=0$, ranging from $-\infty$ to ∞.

② The mean of the t- distribution is $\mu=0$, when $n\geq 2$. The mean is undefind for $n=1$.

The variance of t- distribution is $\sigma^2 = \frac{n}{n-2}$ for $n>2$, and the variance for $n\leq 2$ does not exist.

Proof.

Mean.

By definition $t = \frac{Z}{\sqrt{\frac{U}{n}}}$, where Z is standard normal variable, U is χ^2- random variable and Z and U are independent.

Now $\mu = E(t)$

$$= E\left(\frac{Z}{\sqrt{U/n}}\right)$$

$$= E(Z) \cdot E\left(\frac{1}{\sqrt{\frac{U}{n}}}\right) \quad \text{as Z and U are independent.}$$

$$= 0 \cdot E(\sqrt{n/U}) \quad \because E(Z)=0 \text{ as s.n. variable having mean zero.}$$

$$\Rightarrow \boxed{\mu = 0}$$

Variance.

$$Var(t) = E(t^2) - (E(t))^2 \quad \text{——— ①}$$

Consider,

$$E(t^2) = E\left(\frac{Z^2}{U/n}\right)$$

$$= E(Z^2) \cdot E(n/U) \qquad \because Z \text{ and } U \text{ are indep.}$$

$$= 1 \cdot E(n/U) \qquad \because \text{S. normal variable has unit variance.}$$

$$\Rightarrow E(t^2) = E(n/U)$$

$$= \int_0^{\infty} \frac{n}{u} \cdot \frac{1}{2^{n/2}\,\Gamma(n/2)}\, u^{\frac{n}{2}-1}\, e^{-u/2}\, du$$

$$= n \int_0^{\infty} \frac{1}{2^{n/2}\,\Gamma n/2}\, u^{\frac{n}{2}-2}\, e^{-u/2}\, du$$

$$= \frac{n}{2^{n/2}\,\Gamma n/2} \int_0^{\infty} u^{\frac{n}{2}-2}\, e^{-u/2}\, du$$

Put $\frac{u}{2} = y \Rightarrow u = 2y$

$\Rightarrow du = 2dy$

Then

$$E(t^2) = \frac{n}{2^{n/2}\,\Gamma n/2} \int_0^{\infty} (2y)^{\frac{n}{2}-2}\, e^{-y}\, 2dy$$

$$= \frac{n}{2^{n/2}\,\Gamma n/2} \int_0^{\infty} 2^{\frac{n}{2}-2}\, y^{n/2-2}\, e^{-y}\, 2dy$$

$$= \frac{n}{\cancel{2^{n/2}}\,\Gamma n/2}\, \cancel{2^{n/2}}\, 2^{-2}\, 2 \int_0^{\infty} y^{n/2-2}\, e^{-y}\, dy$$

$$= \frac{n}{2\Gamma(\frac{n}{2})} \int_0^\infty y^{(\frac{n}{2}-1)-1} e^{-y}\, dy$$

$$= \frac{n}{2\Gamma(n/2)} \Gamma(\tfrac{n}{2}-1)$$

$$= \frac{n}{2(\frac{n}{2}-1)\cancel{\Gamma(\frac{n}{2}-1)}} \cancel{\Gamma(\tfrac{n}{2}-1)}$$

$$= \frac{n}{\cancel{2}(\frac{n-2}{\cancel{2}})} = \frac{n}{n-2}$$

Thus eq ① becomes

$$\boxed{Var(t) = \frac{n}{n-2}} \quad \text{for } n<2.$$

③ The t-distribution approaches the standard normal distribution $N(0,1)$ as the number of degrees of freedom n becomes larger. A/2016, A/2018

Proof.

As we know that

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} (1+t^2/n)^{-(n+1)/2}$$

Taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} f(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} (1+t^2/n)^{-(n+1)/2}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} \cdot \lim_{n\to\infty} (1+t^2/n)^{-(n+1)/2} \quad \text{—— ①}$$

Consider,

$$\lim_{n\to\infty}\left(1+t^2/n\right)^{-\left(\frac{n+1}{2}\right)} = \lim_{n\to\infty}\left(1+t^2/n\right)^{-n/2}\left(1+t^2/n\right)^{-1/2}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{t^2}{n}\right)^n\right]^{-1/2} \cdot \lim_{n\to\infty}\left(1+t^2/n\right)^{-1/2}$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^n\right]^{-1/2} (1)$$

$$= \left(e^{t^2}\right)^{-1/2}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2} = e^{-t^2/2} \quad \text{——— (2)}$$

Also

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}$$

$$= \lim_{n\to\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{\pi n}\,\Gamma(n/2)}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n\pi}} \cdot \lim_{n\to\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma(n/2)}$$

Since $\lim_{n\to\infty}\frac{\Gamma(n+k)}{\Gamma(n)} = n^k$

Therefore,

$$\lim_{n\to\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \left(\frac{n}{2}\right)^{1/2}$$

Thus $\lim_{n\to\infty}\frac{1}{\sqrt{n}\,\beta\left(\frac{n}{2},\frac{1}{2}\right)} = \lim_{n\to\infty}\frac{1}{\sqrt{n\pi}}\cdot\sqrt{\frac{n}{2}}$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{2\pi n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} = \frac{1}{\sqrt{2\pi}} \quad \text{———} \; (3)$$

Subsituting (2) and (3) in (1), we get

$$\lim_{n\to\infty} f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

which is the p.d.f of standard normal distribution.

Prove that $\lim_{n\to\infty} \frac{\Gamma(n+k)}{\Gamma(n)} = n^k$.

Proof.

$$L.H.S = \lim_{n\to\infty} \frac{\Gamma(n+k)}{\Gamma(n)} = \lim_{n\to\infty} \frac{(n+k-1)!}{(n-1)!}$$

For very large n

$$n! = \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}$$

So $(n-1)! = \sqrt{2\pi}\, e^{-(n-1)}\, (n-1)^{(n-1)+\frac{1}{2}}$

$$= \sqrt{2\pi}\, e^{-(n-1)}\, (n-1)^{n-\frac{1}{2}}$$

and

$$(n+k-1)! = \sqrt{2\pi}\, e^{-(n+k-1)}\, (n+k-1)^{n+k-\frac{1}{2}}$$

Thus

$$\lim_{n\to\infty} \frac{(n+k-1)!}{(n-1)!} = \lim_{n\to\infty} \frac{\cancel{\sqrt{2\pi}}\, e^{-(n+k-1)}\, (n+k-1)^{n+k-\frac{1}{2}}}{\cancel{\sqrt{2\pi}}\, e^{-(n-1)}\, (n-1)^{n-\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{e^{-k}\, \cancel{e^{-(n-1)}}\, (n+k-1)^{n+k-\frac{1}{2}}}{\cancel{e^{-(n-1)}}\, (n-1)^{n-1/2}}$$

$$= \lim_{n\to\infty} e^{-k} \frac{(n+k-1)^{n+k-\frac{1}{2}}}{(n-1)^{n-1/2}}$$

$$= e^{-k} \lim_{n\to\infty} \frac{(n+k-1)^{k}\,(n+k-1)^{n-1/2}}{(n-1)^{n-1/2}}$$

$$= e^{-k} \lim_{n\to\infty} \frac{\left[n\left(1+\frac{k-1}{n}\right)\right]^{k} \left[n\left(1+\frac{k-1}{n}\right)\right]^{n-1/2}}{\left[n\left(1-\frac{1}{n}\right)\right]^{n-1/2}}$$

$$= e^{-k} \lim_{n\to\infty} \frac{n^{k}\left(1+\frac{k-1}{n}\right)^{k} \cancel{n^{n-1/2}} \left(1+\frac{k-1}{n}\right)^{n-1/2}}{\cancel{n^{n-1/2}}\left(1-1/n\right)^{n-1/2}}$$

$$= e^{-k} \lim_{n\to\infty} \frac{n^{k}\left(1+\frac{k-1}{n}\right)^{k} \cdot \left(1+\frac{k-1}{n}\right)^{n} \cdot \left(1+\frac{k-1}{n}\right)^{-1/2}}{\left(1-\frac{1}{n}\right)^{n}\left(1-\frac{1}{n}\right)^{-1/2}}$$

dundy

$$= e^{-k}\, \underline{n^{k}} \lim_{n\to\infty} \frac{\left[\left(1+\frac{k-1}{n}\right)^{k}\right] \left(1+\frac{k-1}{n}\right)^{n} \left(1+\frac{k-1}{n}\right)^{-1/2}}{\left(1-\frac{1}{n}\right)^{n}\left(1-\frac{1}{n}\right)^{-1/2}}$$

$$= e^{-k}\, n^{k} \lim_{n\to\infty} \frac{\left[\left(1+\frac{k-1}{n}\right)^{k}\right] \left(1+\frac{k-1}{n}\right)^{-1/2} \left(1+\frac{k-1}{n}\right)^{n}}{\left(1-\frac{1}{n}\right)^{n}\left(1-1/n\right)^{-1/2}}$$

$$= e^{-k}\, n^{k} \cdot e^{+k-1} / e^{-1}$$

$$= \frac{e^{-k} n^k \cancel{e^k} \cancel{e^{-1}}}{\cancel{e^{-1}}}$$

$$\Rightarrow \lim_{n \to \infty} \frac{\Gamma(n+k)}{\Gamma(n)} = n^k$$

A/2017, S/2018

Prove that all odd order moments about origin are zero for t-distribution and even oder moments about origin are given by

$$\mu'_{2s} = E(t^{2s}) = \frac{n^s \Gamma(s+\frac{1}{2}) \Gamma(\frac{n}{2}-s)}{\Gamma(\frac{1}{2}) \Gamma(n/2)}$$

Proof.

Odd order moments.

By definition S/2018

$$\mu'_{2s+1} = E(t^{2r+1})$$ but $\mu_{2s+1} = 0$

$$= E\left(\frac{Z}{\sqrt{U/n}}\right)^{2s+1}$$

$$= E(Z^{2s+1}) \cdot E\left(\sqrt{\frac{n}{U}}\right)^{2s+1}$$ as Z and U are indep.

Consider

$$E(Z^{2s+1}) = \int_{-\infty}^{\infty} z^{2s+1} f(z) dz$$

$$= \int_{-\infty}^{\infty} z^{2s+1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2s+1} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} 2(0)$$ odd function.

$\Rightarrow E(Z^{2r+1}) = 0$

Thus

$$\boxed{\mu'_{2r+1} = 0}$$

A/2015

Even order Moments.

$$\mu'_{2r} = E(t^{2r})$$

$$= E\left(\frac{Z}{\sqrt{U/n}}\right)^{2r}$$

$$= E(Z)^{2r} \cdot E\left(\sqrt{\frac{n}{U}}\right)^{2r} \longrightarrow ① \quad \text{as } Z \text{ and } U \text{ are indep.}$$

Consider

$$E(Z)^{2r} = \int_{-\infty}^{\infty} z^{2r} f(z)\, dz$$

$$= \int_{-\infty}^{\infty} z^{2r} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^{2r} e^{-z^2/2}\, dz \longrightarrow Ⓐ \quad \text{(even function)}$$

Put $z^2/2 = y \Rightarrow z^2 = 2y$

$\Rightarrow z = \sqrt{2y} \Rightarrow dz = \frac{1}{\sqrt{2y}}\, dy$

Then

$$E(z^{2r}) = \frac{\cancel{2}}{\sqrt{\cancel{2}\pi}} \int_0^{\infty} (\sqrt{2y})^{2r} e^{-y} \frac{dy}{\sqrt{\cancel{2}y}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} (2y)^{r} e^{-y} y^{-1/2}\, dy$$

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$$= \frac{2^1}{\sqrt{\pi}} \int_0^\infty y^{1-1/2} e^{-y} \, dy$$

$$= \frac{2^1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{(1+\frac{1}{2})-1} \, dy$$

$$= \frac{2^1}{\sqrt{\pi}} \Gamma\left(1+\frac{1}{2}\right) \quad \text{———} \quad (2)$$

Now,

$$E\left(\frac{n}{U}\right)^1 = \int_{-\infty}^{\infty} \left(\frac{n}{u}\right)^1 f(u) \, du$$

$$= \int_0^\infty \left(\frac{n}{u}\right)^1 \frac{1}{2^{n/2}\Gamma(n/2)} u^{n/2-1} e^{-u/2} \, du$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty n^1 u^{-1} u^{n/2-1} e^{-u/2} \, du$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} n^1 \int_0^\infty u^{\frac{n}{2}-1-1} e^{-u/2} \, du$$

Put

$$u/2 = y \Rightarrow u = 2y \Rightarrow du = 2dy$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} n^1 \int_0^\infty (2y)^{\frac{n}{2}-1-1} e^{-y} \, 2dy$$

$$= \frac{1}{\cancel{2^{n/2}}\Gamma(n/2)} n^1 \int_0^\infty \cancel{2^{n/2}} \, 2^{-1} \, \cancel{2^{-1}} \, y^{\frac{n}{2}-1-1} e^{-y} \, \cancel{2} dy$$

$$= \frac{n^r}{2^r \Gamma(n/2)} \int_0^\infty e^{-y} y^{\left(\frac{n}{2}-r\right)-1} dy$$

$$= \frac{n^r}{2^r \Gamma(n/2)} \cdot \Gamma\left(\frac{n}{2}-r\right) \quad \text{——— (3)}$$

Subsituting (2) and (3), in (1), we get

$$\mu'_{2r} = \frac{\cancel{2^r}\,\Gamma(r+\frac{1}{2})}{\sqrt{\pi}} \cdot \frac{n^r}{\cancel{2^r}\,\Gamma(\frac{n}{2})} \Gamma\left(\frac{n}{2}-r\right)$$

$$= \frac{n^r \Gamma(r+\frac{1}{2}) \Gamma(\frac{n}{2}-r)}{\Gamma(\frac{1}{2}) \Gamma(n/2)} \qquad \because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Note.

Since mean $\mu = 0$, therefore

$E(t-\mu)^r = E(t)^r$

$\Rightarrow$ moments about mean

= moments about origin.

Thus

A/2014 Derive μ_{2r} for t-dis.

$$\mu_{2r+1} = 0$$

and $$\mu_{2r} = \frac{n^r \Gamma(r+\frac{1}{2}) \Gamma(\frac{n}{2}-r)}{\Gamma(\frac{1}{2}) \Gamma(n/2)}$$

$$\mu_2 = \frac{n^1 \Gamma(1+\frac{1}{2}) \Gamma(\frac{n}{2}-1)}{\Gamma(\frac{1}{2}) \Gamma(n/2)}$$

$$= \frac{n \frac{1}{2} \cancel{\Gamma(\frac{1}{2})} \cancel{\Gamma(\frac{n}{2}-1)}}{(\frac{n}{2}-1) \cancel{\Gamma(\frac{n}{2}-1)} \cancel{\Gamma(\frac{1}{2})}}$$

$$= \frac{n/2}{(n/2 - 1)} = \frac{n/2}{(n-2)/2} = \frac{n}{n-2}$$

$$\mu_4 = \frac{n^2 \Gamma(2+\frac{1}{2}) \Gamma(\frac{n}{2}-2)}{\Gamma(1/2)\ \Gamma(n/2)}$$

$$= \frac{n^2 (1+\frac{1}{2})(\frac{1}{2}) \cancel{\Gamma(\frac{1}{2})} \cancel{\Gamma(n/2 - 2)}}{(\frac{n}{2}-1)(\frac{n}{2}-2) \cancel{\Gamma(\frac{n}{2}-2)} \cancel{\Gamma(1/2)}}$$

$$= \frac{n^2 (3/2)(1/2)}{(\frac{n}{2}-1)(\frac{n}{2}-2)} = \frac{3n^2/\cancel{4}}{\frac{(n-2)(n-4)}{\cancel{4}}}$$

$$\Rightarrow \mu_4 = \frac{3n^2}{(n-2)(n-4)}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \qquad \because \mu_3 = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\cancel{n^2}}{\cancel{(n-2)}(n-4)} \cdot \frac{(n-2)^2}{\cancel{n^2}}$$

$$= \frac{3(n-2)}{n-4}$$

Recurrence Relation.

We know that

$$\mu_{2r} = \frac{n^r \Gamma(r+\frac{1}{2})\, \Gamma(\frac{n}{2}-r)}{\Gamma(\frac{1}{2})\, \Gamma(n/2)} \quad \text{--- ①}$$

Replace r by $r-1$, we have.

$$\mu_{2r-2} = \frac{n^{r-1} \Gamma(r-\frac{1}{2})\, \Gamma(\frac{n}{2}-r+1)}{\Gamma(1/2)\, \Gamma(n/2)} \quad \text{--- ②}$$

Dividing ① by ②, we get

$$\frac{\mu_{2r}}{\mu_{2r-2}} = \frac{n^r \Gamma(r+1/2)\, \Gamma(n/2-r)}{\cancel{\Gamma(1/2)\, \Gamma(n/2)}} \cdot \frac{\cancel{\Gamma(\frac{1}{2})\Gamma(n/2)}}{n^{r-1}\Gamma(r-\frac{1}{2})\Gamma(\frac{n}{2}-r+1)}$$

$$= \frac{\cancel{n^r} \Gamma(r+1/2)\, \cancel{\Gamma(\frac{n}{2}-r)}}{\cancel{n^r} n^{-1} \Gamma(r-1/2)\, (\frac{n}{2}-r)\cancel{\Gamma(\frac{n}{2}-r)}}$$

$$= n\, \frac{(r-\frac{1}{2})\cancel{\Gamma(r-1/2)}}{(\frac{n}{2}-r)\cancel{\Gamma(r-1/2)}}$$

$$= \frac{n(r-1/2)}{\frac{n}{2}-r} = \frac{n(2r-1)}{n-2r}$$

$$\Rightarrow \mu_{2r} = \frac{n(2r-1)}{n-2r}\, \mu_{2r-2}$$

Question.

If X is t-distributed random variable with n degrees of freedom, then show that $(1+X^2/n)^{-1}$ has a ~~a~~ beta distribution.

Sol.

Given that $X \sim t_n$

Therefore,

$$f(x) = \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})}\left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} ; -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{n}\,\beta(n/2, 1/2)}\left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} dx$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})}\int_{-\infty}^{\infty}\left(1+x^2/n\right)^{-\left(\frac{n+1}{2}\right)} dx$$

$$= \frac{2}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})}\int_{0}^{\infty}\left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} dx \quad \text{even function}$$

Put $\left(1+\frac{x^2}{n}\right)^{-1} = y$

$$\Rightarrow -\left(1+\frac{x^2}{n}\right)^{-2}\left(\frac{2x\,dx}{n}\right) = dy$$

$$\Rightarrow dx = \frac{-n}{2x}\left(1+\frac{x^2}{n}\right)^{2} dy$$

when $x \to 0$, $y \to 1$

$x \to \infty$, $y \to 0$

$$= \frac{2}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} \int_1^0 y^{\frac{n+1}{2}} \left\{ \frac{-n}{2x} \left(1+\frac{x^2}{n}\right)^2 \right\} dy$$

Since $\left(1+\frac{x^2}{n}\right)^{-1} = y$

$$\Rightarrow 1 + x^2/n = 1/y$$

$$\Rightarrow \frac{x^2}{n} = \frac{1}{y} - 1$$

$$\Rightarrow x^2 = n\left(1/y - 1\right)$$

$$\Rightarrow x = \sqrt{n\left(\frac{1-y}{y}\right)}$$

So

$$= \frac{2}{\sqrt{n}\,\beta(\frac{n}{2},\frac{1}{2})} \int_1^0 y^{\frac{n+1}{2}} \left\{ \frac{-n}{2} \sqrt{\frac{y}{n(1-y)}} \left(\frac{1}{y}\right)^2 \right\} dy$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{n}}\,\beta(\frac{n}{2},\frac{1}{2})} \int_0^1 y^{\frac{n+1}{2}} \cancel{\frac{\sqrt{n}}{2}} \sqrt{\frac{y}{1-y}} \, \frac{1}{y^2} \, dy$$

$$= \frac{1}{\beta(\frac{n}{2},\frac{1}{2})} \int_0^1 y^{\frac{n+1}{2}} \, y^{1/2} \, y^{-2} \, \frac{dy}{(1-y)^{1/2}}$$

$$= \frac{1}{\beta(\frac{n}{2},\frac{1}{2})} \int_0^1 y^{\frac{n}{2}+\frac{1}{2}+\frac{1}{2}-2} \, \frac{dy}{(1-y)^{1/2}}$$

$$= \frac{1}{\beta(n/2, 1/2)} \int_0^1 y^{\frac{n}{2}-1} (1-y)^{-1/2} \, dy$$

$$= \frac{1}{\beta(n/2, \frac{1}{2})} \int_0^1 y^{\frac{n}{2}-1} (1-y)^{\frac{1}{2}-1} \, dy$$

$$\Rightarrow f(y) = \frac{1}{\beta(\frac{n}{2}, \frac{1}{2})} \cdot y^{\frac{n}{2}-1} (1-y)^{\frac{1}{2}-1}$$

which is p.d.f. of beta distribution.

The F-Distribution. A/2009

Let s_1^2 and s_2^2 be the unbiased estimated variances of two random samples of sizes n_1 and n_2, drawn from normal populations with same variances. Then the ratio

$$F = \frac{s_1^2}{s_2^2}$$ may be written as

A/2017 Define F-statistic

$$F = \frac{s_1^2}{s_2^2} = \frac{s_1^2/\sigma^2}{s_2^2/\sigma^2}$$

$$\Rightarrow F = \frac{U/\nu_1}{V/\nu_2} = \frac{U\nu_2}{V\nu_1}$$

where

$U = \frac{(n_1-1)s_1^2}{\sigma^2}$ is a χ^2 - variable with $\nu_1 = n_1 - 1$ deg. of freedom

and $V = \frac{(n_2-1)s_2^2}{\sigma^2}$ is a χ^2 - variable with $\nu_2 = n_2 - 1$ deg. of freedom

Derivation.

Let U and V be independent χ^2-variables with ν_1 and ν_2 degrees of freedom resp.

Then their joint distribution is

$$f(u,v) = g(u) \cdot h(v)$$

$$= \frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} u^{\frac{\nu_1}{2}-1} e^{-u/2} \cdot \frac{1}{2^{\nu_2/2}\Gamma(\frac{\nu_2}{2})} v^{\frac{\nu_2}{2}-1} e^{-v/2}$$

$$\Rightarrow f(u,v) = \frac{u^{\frac{\nu_1}{2}-1} v^{\frac{\nu_2}{2}-1} e^{-\frac{1}{2}(u+v)}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}$$

Let us introduce a new variable as $F = \frac{u/\nu_1}{v/\nu_2}$ and $v = z$

so that $F = \frac{u/\nu_1}{z/\nu_2} = \frac{u\nu_2}{z\nu_1}$

$$\Rightarrow u = \frac{z\nu_1}{\nu_2} F, \quad z = v$$

and the Jacobian of transformation is

$$J = \frac{\partial(u,v)}{\partial(F,z)} = \begin{vmatrix} \frac{\partial u}{\partial F} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial F} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{z\nu_1}{\nu_2} & \frac{\nu_1}{\nu_2}F \\ 0 & 1 \end{vmatrix}$$

$$= z\,\nu_1/\nu_2$$

Subsituting these values, we get

$$\Rightarrow f(F,z) = \frac{\left(z\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1}{2}-1} z^{\frac{\nu_2}{2}-1} e^{-\frac{1}{2}\left(\frac{z\nu_1 F}{\nu_2}+z\right)}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}|J|$$

$$= \frac{z^{\frac{\nu_1}{2}-1}\left(\nu_1/\nu_2\right)^{\frac{\nu_1}{2}-1} F^{\frac{\nu_1}{2}-1} z^{\frac{\nu_2}{2}-1} e^{-\left(\frac{\nu_1}{\nu_2}F+1\right)z/2}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}|J|$$

$$= \frac{z^{\frac{\nu_1}{2}+\frac{\nu_2}{2}-2}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}-1} F^{\frac{\nu_1}{2}-1} e^{-\left(\frac{\nu_1}{\nu_2}F+1\right)\frac{z}{2}}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} z\frac{\nu_1}{\nu_2}$$

$$= \frac{z^{\frac{\nu_1}{2}+\frac{\nu_2}{2}-1}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} F^{\frac{\nu_1}{2}-1} e^{-\frac{z}{2}\left(\frac{\nu_1}{\nu_2}F+1\right)}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

Integrating w.r.t. z, we find the density of F as

$$f(F) = \int_0^\infty f(F,z)\,dz$$

$$= \int_0^\infty \frac{z^{(\nu_1+\nu_2)/2-1}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} F^{\frac{\nu_1}{2}-1} e^{-\frac{z}{2}\left(\frac{\nu_1}{\nu_2}F+1\right)}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}dz$$

$$= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} F^{\frac{\nu_1}{2}-1}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\int_0^\infty z^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{z}{2}\left(\frac{\nu_1}{\nu_2}F+1\right)}dz$$

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Put $\frac{z}{2}\left(\frac{\nu_1}{\nu_2}F+1\right) = y$

$$\Rightarrow z = \left(\frac{\nu_1}{\nu_2}F+1\right)^{-1} 2y$$

$$\Rightarrow dz = \left(\frac{\nu_1}{\nu_2}F+1\right)^{-1} 2\,dy$$

Then

$$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}\, F^{\frac{\nu_1}{2}-1}}{2^{\frac{1}{2}(\nu_1+\nu_2)}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty \left[2\left(\frac{\nu_1}{\nu_2}F+1\right)^{-1} y\right]^{\frac{\nu_1+\nu_2}{2}-1} e^{-y}\, 2\left(\frac{\nu_1}{\nu_2}F+1\right)^{-1} dy$$

$$= \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}\, F^{\frac{\nu_1}{2}-1}}{\cancel{2^{\frac{1}{2}(\nu_1+\nu_2)}}\,\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty \cancel{2^{\frac{1}{2}(\nu_1+\nu_2)}} \left(\frac{\nu_1}{\nu_2}F+1\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} e^{-y}\,dy \times y^{\frac{\nu_1+\nu_2}{2}-1}$$

$$= \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}\, F^{\frac{\nu_1}{2}-1}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}F+1\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} \int_0^\infty y^{\frac{\nu_1+\nu_2}{2}-1} e^{-y}\,dy$$

$$= \frac{(\nu_1/\nu_2)^{\nu_1/2}\, F^{\frac{\nu_1}{2}-1}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}F+1\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} \Gamma\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)$$

$$= \frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} F^{\frac{\nu_1}{2}-1} \left(1+\frac{\nu_1}{\nu_2}F\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)}$$

$$\Rightarrow f(F) = \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(\nu_1/\nu_2)^{\nu_1/2} F^{\frac{\nu_1}{2}-1}}{(1+\frac{\nu_1}{\nu_2}F)^{(\nu_1+\nu_2)/2}} \quad ; 0 < F < \infty$$

which is the required F- distribution with ν_1 degrees of freedom in the numerator and ν_2 degrees of freedom in the denominator.

It is abbreviated as $F(\nu_1, \nu_2)$.

It is to be noted that the F-distribution does not depend upon the population variance σ^2 but depends upon the two parameters ν_1 and ν_2 only.

Properties:

① The F-distribution always ranges from zero to plus infinity.

② The mean and variance of the F-distribution with ν_1 and ν_2 degrees of freedom are

A/2008
A/2012

$$\mu = \frac{\nu_2}{\nu_2 - 2} \quad \text{for } \nu_2 > 2$$

and $$\sigma^2 = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} \quad \text{for } \nu_2 > 4$$

Proof.

Mean. S/2018

The mean of the r.v. F, defined as

$F = \frac{U/\nu_1}{V/\nu_2} = \frac{U\nu_2}{V\nu_1}$, wher U and V

are independent chi-square variables with ν_1 and ν_2 degrees of freedom resp, is given by

$$\mu = E(F) = E\left(\frac{U\nu_2}{V\nu_1}\right) = \frac{\nu_2}{\nu_1} E\left(\frac{U}{V}\right)$$

$$= \frac{\nu_2}{\nu_1} E(U) \cdot E\left(\frac{1}{V}\right) \quad \text{——— (1)}$$

$E(U) = \nu_1$ ——— (2) $\because$ ν_1 is degree of freedom of U.

μ = mean of χ^2 is its deg. of freedom.

and

$$E\left(\frac{1}{V}\right) = \int_{-\infty}^{\infty} \frac{1}{V} f(v)\, dv$$

$$= \int_0^{\infty} \frac{1}{V} \cdot \frac{1}{2^{\nu_2/2}\Gamma\left(\frac{\nu_2}{2}\right)} V^{\frac{\nu_2}{2}-1} e^{-V/2}\, dv$$

$$= \frac{1}{2^{\nu_2/2}\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^{\infty} \frac{1}{V} V^{\frac{\nu_2}{2}-1} e^{-\frac{V}{2}}\, dv$$

$$= \frac{1}{2^{\nu_2/2}\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^{\infty} V^{\frac{\nu_2}{2}-2} e^{-V/2}\, dv$$

Put $y = V/2 \Rightarrow V = 2y$

$\Rightarrow dv = 2dy$

$$= \frac{1}{2^{\nu_2/2}\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^{\infty} (2y)^{\frac{\nu_2}{2}-2} e^{-y}\, 2dy$$

$$= \frac{1}{2^{\nu_2/2}\Gamma(\frac{\nu_2}{2})}\int_0^\infty 2^{\frac{\nu_2}{2}-2}\, 2\, y^{\frac{\nu_2}{2}-2} e^{-y}\,dy$$

$$= \frac{1}{\cancel{2^{\nu_2/2}}\Gamma(\frac{\nu_2}{2})}\cancel{2^{\frac{\nu_2}{2}}}\cdot 2^{-1}\int_0^\infty y^{\frac{\nu_2}{2}-2} e^{-y}\,dy$$

$$= \frac{1}{2\Gamma(\nu_2/2)}\int_0^\infty y^{(\frac{\nu_2}{2}-1)-1} e^{-y}\,dy$$

$$= \frac{1}{2\Gamma(\frac{\nu_2}{2})}\Gamma\left(\frac{\nu_2}{2}-1\right)$$

$$= \frac{1}{2(\frac{\nu_2}{2}-1)\cancel{\Gamma(\frac{\nu_2}{2}-1)}}\cdot\cancel{\Gamma\left(\frac{\nu_2}{2}-1\right)}$$

$$= \frac{1}{\cancel{2}\left(\frac{\nu_2-2}{\cancel{2}}\right)} = \frac{1}{\nu_2-2} \quad \text{———} (3)$$

Now equation (1) becomes

(2) and (3) in (1) $\Rightarrow$ $\mu = \frac{\nu_2}{\cancel{\nu_1}}\cancel{\nu_1}\cdot\frac{1}{\nu_2-2}$

$$\Rightarrow \boxed{\mu = \frac{\nu_2}{\nu_2-2}}$$

A/2012, 2009, 2008, 2007

Variance.

$$Var(F) = E(F^2) - (E(F))^2 \quad \text{———} (1)$$

Consider

$$E(F^2) = E\left(\frac{U\nu_2}{V\nu_1}\right)^2 = \frac{\nu_2^2}{\nu_1^2} E\left(U/V\right)^2$$

$$= \frac{\nu_2^2}{\nu_1^2} E(U^2) \cdot E\left(\frac{1}{V^2}\right) \quad \text{———} \quad (*)$$

$E(U^2) =$ ~~$2\nu_1$~~ ?

As $\quad Var(U) = E(U^2) - (E(U))^2$

$\Rightarrow \quad E(U^2) = Var(U) + (E(U))^2$

$\because \quad Var(U) = 2\nu_1$ i.e. 2 times parameter deg. of freedom

and $E(U) = \nu_1$ i.e. equal to its parameter of U. degree of freedom

So

$$E(U^2) = 2\nu_1 + \nu_1^2 \quad \text{———} \quad (**)$$

Now

$$E\left(\frac{1}{V^2}\right) = \int_{-\infty}^{\infty} \frac{1}{V^2} f(v)\,dv$$

$$= \int_0^{\infty} \frac{1}{V^2} \frac{1}{2^{\nu_2/2}\Gamma(\frac{\nu_2}{2})} v^{\frac{\nu_2}{2}-1} e^{-v/2}\,dv$$

$$= \frac{1}{2^{\frac{\nu_2}{2}}\Gamma(\frac{\nu_2}{2})} \int_0^{\infty} V^{\frac{\nu_2}{2}-3} e^{-\frac{V}{2}}\,dV$$

Put $\quad V/2 = y \Rightarrow V = 2y \Rightarrow dv = 2dy$

Then

$$E\left(\frac{1}{V^2}\right) = \frac{1}{2^{\nu_2/2}\Gamma(\frac{\nu_2}{2})} \int_0^{\infty} (2y)^{\frac{\nu_2}{2}-3} e^{-y}\, 2dy$$

$$= \frac{1}{\cancel{2^{\frac{\nu_2}{2}}}\,\Gamma(\frac{\nu_2}{2})} \int_0^{\infty} \cancel{2^{\frac{\nu_2}{2}}}\, 2^{-3}\, y^{\frac{\nu_2}{2}-3} e^{-y}\, 2dy$$

$$= \frac{1}{2^2\,\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty y^{\left(\frac{\nu_2}{2}-2\right)-1} e^{-y}\, dy$$

$$= \frac{1}{4\,\Gamma\left(\frac{\nu_2}{2}\right)} \Gamma\left(\frac{\nu_2}{2}-2\right)$$

$$= \frac{1}{4\left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-2\right)}} \times \cancel{\Gamma\left(\frac{\nu_2}{2}-2\right)}$$

$$= \frac{1}{(\nu_2-2)(\nu_2-4)} \longrightarrow (***)$$

Subsituting (**) and (***) in (*), we get

$$E(F^2) = \frac{2\nu_1+\nu_1^2}{(\nu_2-2)(\nu_2-4)} \cdot \frac{\nu_2^2}{\nu_1^2}$$

Thus equation (1) becomes,

$$Var(F) = \frac{\nu_2^2(2\nu_1+\nu_1^2)}{\nu_1^2(\nu_2-2)(\nu_2-4)} - \frac{\nu_2^2}{(\nu_2-2)^2}$$

$$= \frac{(2\nu_1+\nu_1^2)(\nu_2-2)\nu_2^2 - \nu_2^2(\nu_2-4)\nu_1^2}{\nu_1^2(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{(2\nu_1\nu_2^2+\nu_1^2\nu_2^2)(\nu_2-2) - \nu_1^2\nu_2^2(\nu_2-4)}{\nu_1^2(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{2\nu_1\nu_2^3 - 4\nu_1\nu_2^2 + \cancel{\nu_1^2\nu_2^3} - 2\nu_1^2\nu_2^2 - \cancel{\nu_1^2\nu_2^3} + 4\nu_1^2\nu_2^2}{\nu_1^2(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{2\nu_1\nu_2^3 - 4\nu_1\nu_2^2 + 2\nu_1^2\nu_2^2}{\nu_1^2(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{\cancel{\nu_1}\{2\nu_2^3 - 4\nu_2^2 + 2\nu_1\nu_2^2\}}{\nu_1^{\cancel{2}}(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{2\nu_2^2(\nu_2 - 2 + \nu_1)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}.$$

mode
$\rightarrow$ 5/2018

③ The F-distribution for $\nu_1 > 2$, $\nu_2 > 2$ is unimodal and the mode of distribution with $\nu_1 \geq 2$ is at $F = \dfrac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$.

Proof.

As we know that

$$f(F) = \frac{1}{B_1\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} F^{\frac{\nu_1}{2}-1}}{\left(1 + \frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}}$$

Diff. w.r.t. F, we get

اس پر quotient rule کی بجائے direct solve کریں۔

$$\frac{df}{dF} = \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(1+\frac{\nu_1}{\nu_2}F)^{\frac{\nu_1+\nu_2}{2}} (\frac{\nu_1}{2}-1)F^{\frac{\nu_1}{2}-2} - F^{\frac{\nu_1}{2}-1} (\frac{\nu_1+\nu_2}{2})(1+F\frac{\nu_1}{\nu_2})^{\frac{\nu_1+\nu_2}{2}-1}(\frac{\nu_1}{\nu_2})}{(1+\frac{\nu_1}{\nu_2}F)^{\nu_1+\nu_2}}$$

$$= \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(1+\frac{\nu_1}{\nu_2}F)^{\frac{\nu_1+\nu_2}{2}} \left\{(\frac{\nu_1}{2}-1)F^{-1} - (\frac{\nu_1+\nu_2}{2})(1+\frac{\nu_1}{\nu_2}F)^{-1}\frac{\nu_1}{\nu_2}\right\}F^{\frac{\nu_1}{2}-1}}{(1+\frac{\nu_1}{\nu_2}F)^{\nu_1+\nu_2}}$$

Put $\frac{df}{dF} = 0$

Then $$\frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(1+\nu_1/\nu_2 F)^{\frac{\nu_1+\nu_2}{2}} F^{\frac{\nu_1}{2}-1}\left\{(\frac{\nu_1}{2}-1)F^{-1} - \frac{\nu_1}{\nu_2}(\frac{\nu_1+\nu_2}{2})(1+\frac{\nu_1}{\nu_2}F)^{-1}\right\}}{(1+\frac{\nu_1}{\nu_2}F)^{\nu_1+\nu_2}} = 0$$

$$\Rightarrow \frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{F^{\frac{\nu_1}{2}-1}\left\{(\frac{\nu_1}{2}-1)F^{-1} - \frac{\nu_1}{\nu_2}(\frac{\nu_1+\nu_2}{2})(1+\frac{\nu_1}{\nu_2}F)^{-1}\right\}}{(1+\frac{\nu_1}{\nu_2}F)^{\frac{\nu_1+\nu_2}{2}}} = 0$$

$$\Rightarrow (\frac{\nu_1}{2}-1)F^{-1} - \frac{\nu_1}{\nu_2}(\frac{\nu_1+\nu_2}{2})(1+\frac{\nu_1}{\nu_2}F)^{-1} = 0$$

$$\Rightarrow (\frac{\nu_1}{2}-1)F^{-1} = \frac{\nu_1}{\nu_2}(\frac{\nu_1+\nu_2}{2})(1+\frac{\nu_1}{\nu_2}F)^{-1}$$

$$\Rightarrow (\frac{\nu_1}{2}-1)(1+\frac{\nu_1}{\nu_2}F) = \frac{\nu_1}{\nu_2}(\frac{\nu_1+\nu_2}{2})F$$

$$\Rightarrow \cancel{\frac{\nu_1}{2\nu_2}F} + \frac{\nu_1}{2} - 1 - \frac{\nu_1}{\nu_2}F = \cancel{\frac{\nu_1}{2\nu_2}F} + \frac{\nu_1\cancel{\nu_2}}{2\cancel{\nu_2}}F$$

$$\Rightarrow \frac{\nu_1}{2} - 1 - \frac{\nu_1}{\nu_2}F = \frac{\nu_1}{2}F$$

$$\Rightarrow \frac{\nu_1}{2} - 1 = \frac{\nu_1}{2}F + \frac{\nu_1}{\nu_2}F$$

$$\Rightarrow \frac{\nu_1}{2} - 1 = \nu_1 F\left(\frac{1}{2} + \frac{1}{\nu_2}\right)$$

$$\Rightarrow \frac{\nu_1 - 2}{\cancel{2}} = \nu_1 F\left(\frac{\nu_2 + 2}{\cancel{2}\nu_2}\right)$$

$$\Rightarrow \frac{\nu_1 - 2}{\nu_1} = \frac{\nu_2 + 2}{\nu_2}F$$

$$\Rightarrow F = \frac{\nu_1 - 2}{\nu_1} \cdot \frac{\nu_2}{\nu_2 + 2}$$

$$= \frac{\nu_2(\nu_1 - 2)}{\nu_1(\nu_2 + 2)}.$$

Calculation of $\frac{d^2 f}{dF^2}$:

$$\frac{d^2 f}{dF^2} = \frac{d}{dF}\left[\frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{1}{2}(\nu_1+\nu_2)}} \left\{\left(\frac{\nu_1}{2}-1\right)F^{-1} - \frac{1}{2}(\nu_1+\nu_2)\frac{\nu_1}{\nu_2}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-1}\right\}\right]$$

$$= \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{1}{2}(\nu_1+\nu_2)}} \Big[-\left(\frac{\nu_1}{2}-1\right)F^{-2}$$

$$+\frac{1}{2}(\nu_1+\nu_2)\left(\frac{\nu_1}{\nu_2}\right)\left(1+\frac{\nu_1}{\nu_2}F\right)^{-2}\frac{\nu_1}{\nu_2}\Bigg]$$

$$+\frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}\frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{1}{2}(\nu_1+\nu_2)}}\Bigg[\left(\frac{\nu_1}{2}-1\right)F^{-1}$$

$$-\frac{1}{2}(\nu_1+\nu_2)\frac{\nu_1}{\nu_2}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-1}\Bigg]^2 \text{———} \text{ⓐ}$$

Subsituting the value of F as follows, we get

$$\left(\frac{\nu_1}{2}-1\right)F^{-1}-\frac{1}{2}(\nu_1+\nu_2)\left(\frac{\nu_1}{\nu_2}\right)\left(1+\frac{\nu_1}{\nu_2}F\right)^{-1}$$

$$=\cancel{\left(\frac{\nu_1-2}{2}\right)}\frac{\nu_1(\nu_2+2)}{\nu_2\cancel{(\nu_1-2)}}-\frac{1}{2}(\nu_1+\nu_2)\frac{\nu_1}{\nu_2}\cdot$$

$$\left(1+\cancel{\frac{\nu_1}{\nu_2}}\cdot\cancel{\frac{\nu_2}{\nu_1}}\cdot\frac{\nu_1-2}{\nu_2+2}\right)^{-1}$$

$$=\frac{\nu_1}{\nu_2}\left(\frac{\nu_2+2}{2}\right)-\frac{1}{2}\left(\frac{\nu_1}{\nu_2}\right)\left(\frac{\nu_1+\nu_2}{2}\right)\left(\frac{\nu_2+2+\nu_1-2}{\nu_2+2}\right)^{-1}$$

$$=\frac{\nu_1}{\nu_2}\left(\frac{\nu_2+2}{2}\right)-\frac{\nu_1}{\nu_2}\left(\frac{\nu_1+\nu_2}{2}\right)\left(\frac{\nu_1+\nu_2}{\nu_2+2}\right)^{-1}$$

$$=\frac{\nu_1}{\nu_2}\left(\frac{\nu_2+2}{2}\right)-\frac{\nu_1}{\nu_2}\left(\frac{\nu_1+\cancel{\nu_2}}{2}\right)\left(\frac{\nu_2+2}{\cancel{\nu_1+\nu_2}}\right)$$

$$= \frac{\nu_1}{\nu_2}\left(\frac{\nu_2+2}{2}\right) - \frac{\nu_1}{\nu_2}\left(\frac{\nu_2+2}{2}\right) = 0$$

The 2nd term in ⑧ will go to zero when we put the value of F.

Now,

$$-\left(\frac{\nu_1}{2}-1\right)F^{-2} + \frac{1}{2}(\nu_1+\nu_2)\left(\frac{\nu_1}{\nu_2}\right)^2\left(1+\frac{\nu_1}{\nu_2}F\right)^{-2}$$

$$= -\left(\frac{\nu_1-\cancel{2}}{2}\right)\frac{\nu_1^2(\nu_2+2)^2}{\nu_2^2(\cancel{\nu_1-2})^2} + \frac{1}{2}\frac{\nu_1^2}{\nu_2^2}(\nu_1+\nu_2)$$

$$\left(\frac{\nu_2+2}{\nu_1+\nu_1}\right)^2$$

$$= -\left(\frac{\nu_1}{\nu_2}\right)^2\frac{(\nu_2+2)^2}{2(\nu_1-2)} + \left(\frac{\nu_1}{\nu_2}\right)^2\left(\frac{\cancel{\nu_1+\nu_2}}{2}\right)\left(\frac{\nu_2+2}{(\nu_1+\nu_2)}\right)^2$$

$$= -\left(\frac{\nu_1}{\nu_2}\right)^2\frac{(\nu_2+2)^2}{2}\left\{\frac{1}{\nu_1-2} - \frac{1}{\nu_1+\nu_2}\right\}$$

$$= -\left(\frac{\nu_1}{\nu_2}\right)^2\frac{(\nu_2+2)^2}{2}\left\{\frac{\cancel{\nu_1}+\nu_2-\cancel{\nu_1}+2}{(\nu_1-2)(\nu_1+\nu_2)}\right\}$$

$$= -\left(\frac{\nu_1}{\nu_2}\right)^2\frac{(\nu_2+2)^2}{2}\left\{\frac{\nu_2+2}{(\nu_1-2)(\nu_1+\nu_2)}\right\}$$

Thus equation ⑧ becomes

$$\frac{d^2f}{dF^2} = \frac{1}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \left(\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}\right)^{\frac{\nu_1}{2}-1}}{\left(\frac{\nu_1+\nu_2}{\nu_2+2}\right)^{\frac{1}{2}(\nu_1+\nu_2)}} \times$$

$$\left[-\left(\frac{\nu_1}{\nu_2}\right)^2 \left(\frac{\nu_2+2}{2}\right)^2 \left\{\frac{\nu_2+2}{(\nu_1-2)(\nu_1+\nu_2)}\right\}\right] < 0$$

$$\Rightarrow \quad F = \frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$$

is mode of the F distribution.

(4) If F has a F distribution with ν_1 and ν_2 degrees of freedom, then $\frac{1}{F}$ has an F distribution with ν_2 and ν_1 degrees of freedom.

i.e. If $F \sim F(\nu_1, \nu_2)$ then $\frac{1}{F} \sim F(\nu_2, \nu_1)$.

Proof.

As we know that

$$F = \frac{U/\nu_1}{V/\nu_2} = \frac{U\nu_2}{V\nu_1}$$

where

$$U \sim \chi^2_{\nu_1} \quad \text{and} \quad V \sim \chi^2_{\nu_2}$$

so $$\frac{1}{F} = \frac{V\nu_1}{U\nu_2} = \frac{V/\nu_2}{U/\nu_1}$$

$$= F(\nu_2, \nu_1) \quad \text{as required.}$$

Prove that A/2016,

If $T \sim t_\nu$ then $T^2 \sim F(1, \nu)$

Proof.

The p.d.f. for t-distribution is

$$f(t) = \frac{1}{\sqrt{\nu}\, \beta(\frac{\nu}{2}, \frac{1}{2})} \left(1 + t^2/\nu\right)^{-(\nu+1)/2} \quad \text{for } -\infty < t < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{\sqrt{\nu}\, \beta(\frac{\nu}{2}, \frac{1}{2})} \int_{-\infty}^{\infty} \left(1 + t^2/\nu\right)^{-(\nu+1)/2} dt$$ even function

$$\Rightarrow \int_0^\infty f(t)\,dt = \frac{2}{\sqrt{\nu}\,\beta(\frac{\nu}{2},\frac{1}{2})} \int_0^\infty (1+t^2/\nu)^{-(\nu+1)/2}\,dt \quad \text{——} (1)$$

The p.d.f. of F-distribution is

$$f(F) = \frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})} \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}} F^{\frac{\nu_1}{2}-1}}{(1+\frac{\nu_1}{\nu_2}F)^{\frac{\nu_1+\nu_2}{2}}} \quad ; 0 < F < \infty$$

Put $\nu_1 = 1$ and $\nu_2 = \nu$

Then

$$f(F) = \frac{1}{\beta(\frac{1}{2},\frac{\nu}{2})} \frac{(1/\nu)^{1/2} F^{\frac{1}{2}-1}}{(1+\frac{1}{\nu}F)^{\frac{1+\nu}{2}}}$$

$$\Rightarrow \int_0^\infty f(F)\,dF = \frac{1}{\sqrt{\nu}\,\beta(\frac{\nu}{2},\frac{1}{2})} \int_0^\infty \frac{F^{-1/2}}{(1+F/\nu)^{(\nu+1)/2}}\,dF$$

$$= \frac{1}{\sqrt{\nu}\,\beta(\frac{\nu}{2},\frac{1}{2})} \int_0^\infty \frac{(1+F/\nu)^{-(\nu+1)/2}}{\sqrt{F}}\,dF \quad \text{——} (2)$$

Put $t^2 = w$ in (1), we get

$$\int_0^\infty f(t)\,dt = \frac{\cancel{2}}{\sqrt{\nu}\,\beta(\frac{\nu}{2},\frac{1}{2})} \int_0^\infty (1+\frac{w}{\nu})^{-(\nu+1)/2} \frac{dw}{\cancel{2}\sqrt{w}}$$

$$= \frac{1}{\sqrt{\nu}\,\beta(\frac{\nu}{2},\frac{1}{2})} \int_0^\infty \frac{(1+w/\nu)^{-(\nu+1)/2}}{\sqrt{w}}\,dw \longrightarrow (3)$$

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From ① and ③, we have

$W \sim F(1,\nu)$

$\Rightarrow t^2 \sim F(1,\nu)$

Thus, if $T \sim t_{\nu}$ then $T^2 \sim F(1,\nu)$.

Prove that A/2015, A/2017,

If $F \sim F(\nu_1,\nu_2)$ then $Y = (1+\frac{\nu_1}{\nu_2}F)^{-1} \sim \beta(\frac{\nu_2}{2},\frac{\nu_1}{2})$

Proof.

As we know that

$$\int_0^\infty f(F)\,dF = \frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}(\nu_1/\nu_2)^{\nu_1/2}\int_0^\infty \frac{F^{\frac{\nu_1}{2}-1}}{(1+\frac{\nu_1}{\nu_2}F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

Put $(1+\frac{\nu_1}{\nu_2}F)^{-1} = y$

then $1+\nu_1/\nu_2 F = 1/y \Rightarrow \frac{\nu_1}{\nu_2}F = 1/y - 1$

$$\Rightarrow -(1+\frac{\nu_1}{\nu_2}F)^{-2}(\frac{\nu_1}{\nu_2})\,dF = dy$$

when

$F \to 0, \quad y \to 1$

$F \to \infty, \quad y \to 0$

$$\Rightarrow \int_0^\infty f(F)\,dF = \frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}(\nu_1/\nu_2)^{\nu_1/2}\int_1^0 \left[\frac{\nu_2}{\nu_1}(\frac{1}{y}-1)\right]^{\frac{\nu_1}{2}-1}(y)^{\frac{\nu_1+\nu_2}{2}}\left[-\frac{\nu_2}{\nu_1}(1+\frac{\nu_1}{\nu_2}F)^2dy\right]$$

$$= \frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}\cancel{(\nu_1/\nu_2)^{\nu_1/2}}\int_0^1 \cancel{(\frac{\nu_2}{\nu_1})^{\frac{\nu_1}{2}-1}}\,y^{\frac{\nu_1+\nu_2}{2}}\frac{(1-y)^{\frac{\nu_1}{2}-1}}{y^{\frac{\nu_1}{2}-1}}\cancel{\frac{\nu_2}{\nu_1}}\frac{1}{y^2}dy$$

$$= \frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}\int_0^1 \cancel{y^{\frac{\nu_1}{2}}}\cdot y^{\nu_2/2}(1-y)^{\frac{\nu_1}{2}-1}\frac{dy}{\cancel{y^{\frac{\nu_1}{2}}}\,y}$$

$$= \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \int_0^1 y^{\frac{\nu_2}{2}-1} (1-y)^{\frac{\nu_1}{2}-1} \, dy$$

which is p.d.f. of $\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})$.

Hence $y \sim \beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})$.

Moments About Origin.

A/2007.

A/2018

$$\mu_r' = E(F^r)$$

$$= E\left(\frac{U\nu_2}{V\nu_1}\right)^r$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^r E\left(\frac{U^r}{V^r}\right)$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^r E(U^r) \cdot E\left(\frac{1}{V^r}\right) \rightarrow (1) \because U \text{ and } V \text{ are indep}$$

Now,

$$E(U^r) = \int_{0}^{\infty} u^r f(u) \, du$$

$$= \int_0^\infty u^r \frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} u^{\frac{\nu_1}{2}-1} e^{-u/2} \, du$$

$$= \frac{1}{2^{\frac{\nu_1}{2}} \Gamma(\frac{\nu_1}{2})} \int_0^\infty u^{\frac{\nu_1}{2}+r-1} e^{-u/2} \, du$$

Put $u/2 = y \Rightarrow u = 2y$

$\Rightarrow du = 2\,dy$

$$= \frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} \int_0^\infty (2y)^{\frac{\nu_1}{2}+r-1} e^{-y} \, 2dy$$

$$= \frac{1}{\cancel{2^{\frac{\nu_1}{2}}}\,\Gamma\left(\frac{\nu_1}{2}\right)} \int_0^\infty \cancel{2^{\frac{\nu_1}{2}}}\, 2^{s}\, \cancel{2^{-1}}\, y^{\frac{\nu_1}{2}+s-1} e^{-y}\, \cancel{2}\, dy$$

$$= \frac{2^{s}}{\Gamma(\nu_1/2)} \int_0^\infty y^{\left(\frac{\nu_1}{2}+s\right)-1} e^{-y}\, dy$$

$$= \frac{2^{s}}{\Gamma\left(\frac{\nu_1}{2}\right)} \Gamma\left(\frac{\nu_1}{2}+s\right) \qquad \text{——— } (2)$$

$$E\left(\frac{1}{V^{s}}\right) = \int_{-\infty}^{\infty} \frac{1}{v^{s}} f(v)\, dv$$

$$= \int_0^\infty \frac{1}{v^{s}} \frac{1}{2^{\nu_2/2}\,\Gamma\left(\frac{\nu_2}{2}\right)} v^{\frac{\nu_2}{2}-1} e^{-v/2}\, dv$$

$$= \frac{1}{2^{\nu_2/2}\,\Gamma(\nu_2/2)} \int_0^\infty v^{\frac{\nu_2}{2}-s-1} e^{-v/2}\, dv$$

Put $v/2 = w \Rightarrow v = 2w \Rightarrow dv = 2dw$

$$= \frac{1}{2^{\frac{\nu_2}{2}}\,\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty (2w)^{\frac{\nu_2}{2}-s-1} e^{-w}\, 2dw$$

$$= \frac{1}{\cancel{2^{\nu_2/2}}\,\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty \cancel{2^{\frac{\nu_2}{2}}}\, 2^{-s}\, 2^{-1}\, \cancel{2}\, w^{\frac{\nu_2}{2}-s-1} e^{-w}\, \cancel{2}\, dw$$

$$= \frac{1}{2^{s}\,\Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty w^{\left(\frac{\nu_2}{2}-s\right)-1} e^{-w}\, dw$$

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$$= \frac{1}{2^{1}\Gamma\left(\frac{\nu_2}{2}\right)} \Gamma\left(\frac{\nu_2}{2}-1\right) \quad \text{———— (3)}$$

Subsituting (2) and (3) in (1), we get

$$\mu_1' = \left(\frac{\nu_2}{\nu_1}\right)^{1} \frac{\cancel{2}^{1}}{\Gamma\left(\frac{\nu_1}{2}\right)} \Gamma\left(\frac{\nu_1}{2}+1\right) \frac{1}{\cancel{2}^{1}\Gamma\left(\frac{\nu_2}{2}\right)} \Gamma\left(\frac{\nu_2}{2}-1\right)$$

$$= \frac{\nu_2^{1}\, \Gamma\left(\frac{\nu_1}{2}+1\right) \Gamma\left(\frac{\nu_2}{2}-1\right)}{\nu_1^{1}\, \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)}$$

$$\Rightarrow \mu_1' = \left(\frac{\nu_2}{\nu_1}\right) \frac{\Gamma\left(\frac{\nu_1}{2}+1\right)\Gamma\left(\frac{\nu_2}{2}-1\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$= \frac{\nu_2}{\nu_1} \frac{\frac{\nu_1}{2}\cancel{\Gamma\left(\nu_1/2\right)}\, \cancel{\Gamma\left(\frac{\nu_2}{2}-1\right)}}{\left(\frac{\nu_2}{2}-1\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-1\right)}\, \cancel{\Gamma\left(\frac{\nu_1}{2}\right)}}$$

$$= \frac{\nu_2}{\nu_1} \frac{\nu_1/2}{\frac{\nu_2}{2}-1} = \frac{\nu_2}{\cancel{\nu_1}} \frac{\cancel{\nu_1}}{\nu_2-2}$$

$$\Rightarrow \mu_1' = \frac{\nu_2}{\nu_2-2}$$

$$\mu_2' = \left(\frac{\nu_2}{\nu_1}\right)^{2} \frac{\Gamma\left(\frac{\nu_1}{2}+2\right)\Gamma\left(\frac{\nu_2}{2}-2\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\nu_2/2\right)}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\left(\frac{\nu_1}{2}+1\right)\frac{\nu_1}{2}\,\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}\,\cancel{\Gamma\left(\frac{\nu_2}{2}-2\right)}}{\left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-2\right)}\,\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{(\nu_1+2)\,\nu_1/2}{(\nu_2-2)\frac{(\nu_2-4)}{2}}$$

$$= \frac{\nu_2^2(\nu_1+2)\cancel{\nu_1}}{\nu_1^{\cancel{2}}(\nu_2-2)(\nu_2-4)} = \frac{\nu_2^2\cdot(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}$$

$$\mu_3' = \left(\frac{\nu_2}{\nu_1}\right)^3 \frac{\Gamma\left(\frac{\nu_1}{2}+3\right)\Gamma\left(\frac{\nu_2}{2}-3\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^3 \frac{\left(\frac{\nu_1}{2}+2\right)\left(\frac{\nu_1}{2}+1\right)\frac{\nu_1}{2}\,\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}\,\cancel{\Gamma\left(\frac{\nu_2}{2}-3\right)}}{\left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right)\left(\frac{\nu_2}{2}-3\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-3\right)}\,\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^3 \frac{(\nu_1+4)(\nu_1+2)\,\nu_1}{(\nu_2-2)(\nu_2-4)(\nu_2-6)}$$

$$= \frac{\nu_2^3(\nu_1+4)(\nu_1+2)}{\nu_1^2(\nu_2-2)(\nu_2-4)(\nu_2-6)}$$

$$\mu_4' = \left(\frac{\nu_2}{\nu_1}\right)^4 \frac{\Gamma\left(\frac{\nu_1}{2}+4\right)\Gamma\left(\frac{\nu_2}{2}-4\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^4 \frac{\left(\frac{\nu_1}{2}+3\right)\left(\frac{\nu_1}{2}+2\right)\left(\frac{\nu_1}{2}+1\right)\frac{\nu_1}{2}\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}\cancel{\Gamma\left(\frac{\nu_2}{2}-4\right)}}{\left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right)\left(\frac{\nu_2}{2}-3\right)\left(\frac{\nu_2}{2}-4\right)\cancel{\Gamma\left(\frac{\nu_2}{2}-4\right)}\cancel{\Gamma\left(\frac{\nu_1}{2}\right)}}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^4 \frac{(\nu_1+6)(\nu_1+4)(\nu_1+2)\,\nu_1}{(\nu_2-2)(\nu_2-4)(\nu_2-6)(\nu_2-8)}$$

$$=\frac{\nu_2^4\,(\nu_1+2)(\nu_1+4)(\nu_1+6)}{\nu_1^3\,(\nu_2-2)(\nu_2-4)(\nu_2-6)(\nu_2-8)}$$

Find moments about origin using p.d.f of F- distribution.

$$\mu_r' = E(F^r)$$

$$= \int_{-\infty}^{\infty} F^r f(F)\,dF$$

$$= \int_0^{\infty} F^r \frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \frac{(\nu_1/\nu_2)^{\nu_1/2}\, F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}}\,dF$$

$$= \frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \int_0^{\infty} F^{\frac{\nu_1}{2}+r-1}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} dF$$

Put $\left(1+\frac{\nu_1}{\nu_2}F\right)^{-1} = y$

$$\Rightarrow 1+\frac{\nu_1}{\nu_2}F = \frac{1}{y} \quad \Rightarrow \frac{\nu_1}{\nu_2}F = \frac{1}{y}-1$$

when $F \to 0$
$y \to 1$
when $F \to \infty$
$y \to 0$

$$\Rightarrow F = \frac{\nu_2}{\nu_1}\left(\frac{1-y}{y}\right)$$

$$\Rightarrow dy = -\left(1+\frac{\nu_1}{\nu_2}F\right)^{-2}\left(\frac{\nu_1}{\nu_2}\right)dF$$

$$\Rightarrow dy = -(y)^2 \frac{\nu_1}{\nu_2} dF \Rightarrow dF = -y^{-2}\frac{\nu_2}{\nu_1}dy$$

Then

$$\mu_r' = \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \int^{0} \left[\frac{\nu_2}{\nu_1}\left(\frac{1-y}{y}\right)\right]^{\frac{\nu_1}{2}+r-1} y^{\frac{\nu_1+\nu_2}{2}} \left[-y^{-2}\frac{\nu_2}{\nu_1}\right]dy$$

$$= \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \cancel{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}} \int_0^1 \cancel{\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_1}{2}}} \left(\frac{\nu_2}{\nu_1}\right)^{r} \cancel{\left(\frac{\nu_2}{\nu_1}\right)^{-1}} (1-y)^{\frac{\nu_1}{2}+r-1} y^{\frac{\nu_2}{2}-r-1} \cancel{\frac{\nu_2}{\nu_1}} dy$$

$$= \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \int_0^1 \left(\frac{\nu_2}{\nu_1}\right)^{r} y^{(\frac{\nu_2}{2}-r)-1} (1-y)^{(\frac{\nu_1}{2}+r)-1} dy$$

$$= \frac{(\nu_2/\nu_1)^r}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \beta\left(\frac{\nu_2}{2}-r, \frac{\nu_1}{2}+r\right)$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^r \frac{\cancel{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2})}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \cdot \frac{\Gamma(\frac{\nu_2}{2}-r)\Gamma(\frac{\nu_1}{2}+r)}{\cancel{\Gamma(\frac{\nu_2}{2}+\frac{\nu_1}{2})}}$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^r \frac{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2}-r)}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}$$

Question.

Derive χ^2-distribution as a limiting form of F-distribution also find its variance.

Sol.

Let $X \sim F(\nu_1, \nu_2)$

Then $$f(x) = \frac{1}{\beta(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(\nu_1/\nu_2)^{\nu_1/2} x^{\frac{\nu_1}{2}-1}}{(1+\frac{\nu_1}{\nu_2}x)^{(\nu_1+\nu_2)/2}}$$

Since $F = \left(\frac{U/\nu_1}{V/\nu_2}\right)$ distribution is ratio of two χ^2-variates.

We know that F-distribution under the condition $\nu_2 \to \infty$ can be approximated by a $\chi^2_{\nu_1}$-distribution.

Therefore,

$$\lim_{\nu_2 \to \infty} f(x) = \lim_{\nu_2 \to \infty} \frac{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-(\nu_1+\nu_2)/2}$$

$$= \lim_{\nu_2 \to \infty} \frac{(\nu_1)^{\nu_1/2} \Gamma(\frac{\nu_1+\nu_2}{2})}{(\nu_2)^{\nu_1/2} \Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} x^{\frac{\nu_1}{2}-1} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{(\nu_1+\nu_2)}{2}}$$

$$= \frac{(\nu_1)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})} x^{\frac{\nu_1}{2}-1} \lim_{\nu_2 \to \infty} \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_2}{2})(\nu_2)^{\nu_1/2}} \lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{(\nu_1}{2}} \longrightarrow (1)$$

Consider

$$\lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{(\nu_1+\nu_2)}{2}} = \lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\nu_1/2} \lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_2}{2}}$$

$$= (1)\left[\lim_{\nu_2\to\infty}\left(1+\frac{\nu_1 x}{\nu_2}\right)^{\nu_2}\right]^{-1/2}$$

$$= \left(e^{\nu_1 x}\right)^{-1/2} = e^{-\nu_1 x/2}$$

$$\Rightarrow \lim_{\nu_2\to\infty}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)} = e^{-\nu_1 x/2} \quad \text{——— (2)}$$

Now

$$\lim_{\nu_2\to\infty}\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left(\frac{\nu_2}{2}\right)^{\frac{\nu_1}{2}}\Gamma\left(\frac{\nu_2}{2}\right)} = \lim_{\nu_2\to\infty}\frac{\left(\frac{\nu_1+\nu_2}{2}-1\right)!}{\left(\frac{\nu_2}{2}\right)^{\nu_1/2}\left(\frac{\nu_2}{2}-1\right)!} \quad \text{——— (3)}$$

Using Stirling approximation as follows

As $n! = \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}$

so $\left(\frac{\nu_1+\nu_2}{2}-1\right)! = \sqrt{2\pi}\, e^{-\left(\frac{\nu_1+\nu_2}{2}-1\right)}\left(\frac{\nu_1+\nu_2}{2}-1\right)^{\frac{\nu_1+\nu_2}{2}-\frac{1}{2}}$

and $\left(\frac{\nu_2}{2}-1\right)! = \sqrt{2\pi}\, e^{-\left(\frac{\nu_2}{2}-1\right)}\left(\frac{\nu_2}{2}-1\right)^{\frac{\nu_2}{2}-\frac{1}{2}}$

Thus equation (3) becomes

$$\lim_{\nu_2\to\infty}\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left(\frac{\nu_2}{2}\right)^{\frac{\nu_1}{2}}\Gamma\left(\frac{\nu_2}{2}\right)} = \lim_{\nu_2\to\infty}\frac{e^{-\frac{\nu_1}{2}}\cancel{e^{-\left(\frac{\nu_2}{2}-1\right)}}\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}-1\right)^{\frac{\nu_1+\nu_2-1}{2}}}{\cancel{e^{-\left(\frac{\nu_2}{2}-1\right)}}\left(\frac{\nu_2}{2}-1\right)^{\frac{\nu_2-1}{2}}\left(\frac{\nu_2}{2}\right)^{\nu_1/2}}$$

$$= e^{-\frac{\nu_1}{2}}\lim_{\nu_2\to\infty}\frac{\left[\frac{\nu_2}{2}\left(1-\frac{2}{\nu_2}+\frac{\nu_1}{\nu_2}\right)\right]^{\frac{\nu_1+\nu_2-1}{2}}}{\left[\frac{\nu_2}{2}\left(1-\frac{2}{\nu_2}\right)\right]^{\frac{\nu_2-1}{2}}\left(\frac{\nu_2}{2}\right)^{\nu_1/2}}$$

$$= e^{-\frac{\nu_1}{2}}\lim_{\nu_2\to\infty}\frac{\left(\frac{\nu_2}{2}\right)^{\frac{\nu_1+\nu_2-1}{2}}\left(1-\frac{2-\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2-1}{2}}}{\left(\frac{\nu_2}{2}\right)^{\frac{\nu_2-1}{2}}\left(1-\frac{2}{\nu_2}\right)^{\frac{\nu_2-1}{2}}\left(\frac{\nu_2}{2}\right)^{\nu_1/2}}$$

$$= e^{-\frac{\nu_1}{2}} \lim_{\nu_2 \to \infty} \frac{2^{\frac{\nu_2-1}{2}} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{\frac{\nu_1+\nu_2-1}{2}} \cancel{\nu_2^{\frac{\nu_1+\nu_2-1}{2}}}}{2^{\frac{\nu_1+\nu_2-1}{2}} \cancel{\nu_2^{(\nu_1+\nu_2-1)/2}} \left(1-2/\nu_2\right)^{\frac{\nu_2-1}{2}}}$$

$$= e^{-\frac{\nu_1}{2}} \lim_{\nu_2 \to \infty} \frac{\left(1+\frac{\nu_1-2}{\nu_2}\right)^{\nu_1/2} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{\frac{\nu_2}{2}} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{-1/2}}{2^{\nu_1/2} \left(1-2/\nu_2\right)^{\frac{\nu_2}{2}} \left(1-\frac{2}{\nu_2}\right)^{-1/2}}$$

$$= \frac{e^{-\frac{\nu_1}{2}}}{2^{\frac{\nu_1}{2}}} \lim_{\nu_2 \to \infty} \frac{\left(1+\frac{\nu_1-2}{\nu_2}\right)^{\nu_1/2} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{\nu_2/2} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{-1/2}}{\left(1-\frac{2}{\nu_2}\right)^{\nu_2/2} \left(1-\frac{2}{\nu_2}\right)^{-1/2}}$$

$$= \frac{e^{-\frac{\nu_1}{2}}}{2^{\nu_1/2}} \frac{(1) \lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{\nu_2/2} (1)}{\lim_{\nu_2 \to \infty} \left(1-\frac{2}{\nu_2}\right)^{\nu_2/2} (1)}$$

$$= \frac{e^{-\frac{\nu_1}{2}}}{2^{\nu_1/2}} \frac{\left\{\lim_{\nu_2 \to \infty} \left(1+\frac{\nu_1-2}{\nu_2}\right)^{\nu_2}\right\}^{1/2}}{\left\{\lim_{\nu_2 \to \infty} \left(1-\frac{2}{\nu_2}\right)^{\nu_2}\right\}^{1/2}}$$

$$= \frac{e^{-\frac{\nu_1}{2}}}{2^{\frac{\nu_1}{2}}} \left(\frac{e^{\nu_1-2}}{e^{-2}}\right)^{1/2}$$

$$= \frac{\cancel{e^{-\nu_1/2}}}{2^{\nu_1/2}} \frac{\cancel{e^{\frac{\nu_1}{2}}} \cancel{e^{-1}}}{\cancel{e^{-1}}}$$

$$\Rightarrow \lim_{\nu_2 \to \infty} \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{(\nu_2)^{\frac{\nu_1}{2}} \Gamma\left(\frac{\nu_2}{2}\right)} = 2^{-\frac{\nu_1}{2}} \quad \text{———} \quad (4)$$

Subsituting (2) and (4) in (1), we get

$$\lim_{\nu_2 \to \infty} f(x) = 2^{-\frac{\nu_1}{2}} e^{-\nu_1 x/2} \frac{(\nu_1)^{\nu_1/2}}{\Gamma\left(\frac{\nu_1}{2}\right)} x^{\frac{\nu_1}{2}-1}$$

$$= \frac{1}{\Gamma\left(\frac{\nu_1}{2}\right)} 2^{-\frac{\nu_1}{2}} e^{-\frac{\nu_1 x}{2}} \nu_1 (\nu_1 x)^{\frac{\nu_1}{2}-1}$$

$$\Rightarrow \lim_{\nu_2 \to \infty} f(x)\, dx = \frac{1}{2^{\frac{\nu_1}{2}} \Gamma\left(\frac{\nu_1}{2}\right)} (\nu_1 x)^{\frac{\nu_1}{2}-1} e^{-\frac{\nu_1 x}{2}} \nu_1\, dx$$

اس کو کائی Square بنانا ہے

Put $\nu_1 x = y \Rightarrow \nu_1 dx = dy$

Then

$$\lim_{n \to \infty} f(y)\, dy = \frac{1}{2^{\frac{\nu_1}{2}} \Gamma\left(\frac{\nu_1}{2}\right)} y^{\frac{\nu_1}{2}-1} e^{-y/2}\, dy$$

which is p.d.f. of χ^2 – distribution.

and variance of χ^2 is $2\nu_1$ because ν_1 is degrees of freedom of χ^2 – distribution.

Question.

If $X_1, X_2, \cdots, X_n$ are independent and identically distributed gamma variates then what will be the distribution of $X_1 + X_2 + \cdots + X_n$.

Sol.

Given $X_i \sim \text{Gamma}\,(m)$

where m is parameter of Gamma distribution.

Therefore, $f(x) = \dfrac{1}{\Gamma(m)} x^{m-1} e^{-x}$; $0 < x < \infty$

Let $S = X_1 + X_2 + \cdots + X_n$ such that $0 < s < \infty$.

Then

$$M_s(t) = E(e^{ts})$$

$$= E(e^{t(X_1 + X_2 + \cdots + X_n)})$$

$$= E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n})$$

Since $X_1, X_2, \cdots, X_n$ are all independent.

Therefore,

$$M_s(t) = E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n})$$

$$= \prod_{i=1}^{n} E(e^{tX_i})$$

$$= \prod_{i=1}^{n} M_{x_i}(t)$$

$$= \prod_{i=1}^{n} M_x(t)$$

$$\Rightarrow M_s(t) = \left(M_x(t)\right)^n \quad \text{———} \quad (1)$$

Now, $M_x(t) = E(e^{tx})$

$$= \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \frac{1}{\Gamma(m)} e^{-x} x^{m-1}\, dx$$

$$= (1-t)^{-m} \quad t<1$$

already calculated in gamma dis.

Thus equation (1) becomes

$$M_s(t) = (1-t)^{-mn}$$

Thus $S \sim$ Gamma (mn).

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Regression AnD Correlation:

$$b_{xy} = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - (\sum y)^2} \quad \text{——— (A)}$$

and

$$b_{yx} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad \text{——— (B)}$$

Simple Regression Co-efficient.

$$x_1 = a + b_{12}\, x_2 \quad \text{——— (1)}$$

and $x_2 = c + b_{21}\, x_1$ ——— (2)

when

$$\bar{x}_1 = \bar{x}_2 = 0$$

Then

$$(1) \Rightarrow \sum x_1 = a\sum(1) + b_{12}\sum x_2$$

$$\Rightarrow \sum x_1 = na + b_{12}\sum x_2$$

$$\Rightarrow \frac{\sum x_1}{n} = \frac{na}{n} + b_{12}\frac{\sum x_2}{n}$$

$$\Rightarrow \bar{x}_1 = a + b_{12}\bar{x}_2$$

Similarly,

$$(2) \Rightarrow \bar{x}_2 = c + b_{21}\bar{x}_1$$

Since $\bar{x}_1 = \bar{x}_2 = 0$

so $a = 0 = c$

$$\Rightarrow \hat{x}_1 = b_{12}x_2$$

and $\hat{x}_2 = b_{21}x_1$

$\boxed{\text{Multiple Regression Co-efficient}}$

$$x_1 = b_{1.23} + b_{12.3}x_2 + b_{13.2}x_3$$
$$x_2 = b_{2.13} + b_{21.3}x_1 + b_{23.1}x_3$$
$$x_3 = b_{3.12} + b_{31.2}x_1 + b_{32.1}x_2$$

$$\Rightarrow \bar{x}_1 = b_{1.23} + b_{12.3}\bar{x}_2 + b_{13.2}\bar{x}_3$$
$$\bar{x}_2 = b_{2.13} + b_{21.3}\bar{x}_1 + b_{23.1}\bar{x}_3$$
$$\bar{x}_3 = b_{3.12} + b_{31.2}\bar{x}_1 + b_{32.1}\bar{x}_2$$

When $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$

Then

$$b_{1.23} = b_{2.13} = b_{3.12} = 0$$

Thus,

$$\hat{x}_1 = b_{12.3}\, x_2 + b_{13.2}\, x_3$$
$$\hat{x}_2 = b_{21.3}\, x_1 + b_{23.1}\, x_3$$
$$\hat{x}_3 = b_{31.2}\, x_1 + b_{32.1}\, x_2$$

Residual:

Let $x_{1.2}$ and $x_{2.1}$ denote residuals for simple regression and $x_{1.23}$, $x_{2.13}$, $x_{3.12}$ denote residual for multiple regression.

Then,

$$x_{1.2} = x_1 - \hat{x}_1$$
$$= x_1 - b_{12}\, x_2$$

and $x_{2.1} = x_2 - \hat{x}_2$

$$= x_2 - b_{21}\, x_1$$

and similarly,

$$x_{1.23} = x_1 - \hat{x}_1$$
$$= x_1 - b_{12.3}\, x_2 - b_{13.2}\, x_3$$

$$x_{2.13} = x_2 - \hat{x}_2$$

$$= x_2 - b_{21.3}\, x_1 - b_{23.1}\, x_3$$

$x_{3.12} = x_3 - \hat{x}_3$

$= x_3 - b_{31.2}x_1 - b_{32.1}x_2$

Q:- If $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$ then prove the following:

① $\sum x_2 x_{1.23} = \sum x_3 x_{1.23}$

As we know that

$x_{1.23} = x_1 - b_{12.3}x_2 - b_{13.2}x_3$

so

$\sum x_2 x_{1.23} = \sum x_2 (x_1 - b_{12.3}x_2 - b_{13.2}x_3)$

$= \sum x_1 x_2 - b_{12.3}\sum (x_2)^2 - b_{13.2}\sum x_2 x_3$

and,

$\sum x_3 x_{1.23} = \sum x_3 (x_1 - b_{12.3}x_2 - b_{13.2}x_3)$

$= \sum x_1 x_3 - b_{12.3}\sum x_2 x_3 - b_{13.2}\sum (x_3)^2$

Since

$\bar{x}_1 = b_{1.23} + b_{12.3}\bar{x}_2 + b_{13.2}\bar{x}_3$

$\bar{x}_2 = b_{2.13} + b_{21.3}\bar{x}_1 + b_{23.1}\bar{x}_3$

$\bar{x}_3 = b_{3.12} + b_{31.2}\bar{x}_1 + b_{32.1}\bar{x}_2$

Put $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$

$$\Rightarrow b_{1\cdot 23} = b_{2\cdot 13} = b_{3\cdot 12} = 0$$

$$\Rightarrow x_1 = b_{12\cdot 3}\, x_2 + b_{13\cdot 2}\, x_3 \quad \text{——— (1)}$$

$$x_2 = b_{21\cdot 3}\, x_1 + b_{23\cdot 1}\, x_3 \quad \text{——— (2)}$$

$$x_3 = b_{31\cdot 2}\, x_1 + b_{32\cdot 1}\, x_2 \quad \text{——— (3)}$$

1st normal equation of (1) is

$$\sum x_1 = b_{12\cdot 3} \sum x_2 + b_{13\cdot 2} \sum x_3 \quad \text{——— (i)}$$

2nd normal equation of (1) is

$$\sum x_1 x_2 = b_{12\cdot 3} \sum (x_2)^2 + b_{13\cdot 2} \sum x_2 x_3$$

$$\Rightarrow \sum x_1 x_2 - b_{12\cdot 3} \sum x_2^2 - b_{13\cdot 2} \sum x_2 x_3 = 0 \quad \text{——— (ii)}$$

3rd normal equation of (1) is

$$\sum x_1 x_3 = b_{12\cdot 3} \sum x_1 x_2 + b_{13\cdot 2} \sum x_3^2$$

$$\Rightarrow \sum x_1 x_3 - b_{12\cdot 3} \sum x_1 x_2 - b_{13\cdot 2} \sum x_3^2 = 0 \quad \text{——— (iii)}$$

~~Hence from (ii) and (iii), we have~~
$\cancel{\sum x_2 x_{1\cdot 23} = \sum x_3 x_{1\cdot 23}}$

Hence from (ii) and (iii), we have

$$\sum x_2\, x_{1\cdot 23} = \sum x_3\, x_{1\cdot 23}.$$

(2) $\sum x_{2.1}\, x_{2.13} = \sum x_2\, x_{2.13}$

AS $x_{2.1} = x_2 - b_{21} x_1$

and

$x_{2.13} = x_2 - b_{21.3} x_1 - b_{23.1} x_3$

$$\sum x_{2.1}\, x_{2.13} = \sum (x_2 - b_{21} x_1)(x_2 - b_{21.3} x_1 - b_{23.1} x_3)$$

$$= \sum (x_2 - b_{21} x_1)\, x_{2.13}$$

$$= \sum x_2\, x_{2.13} - b_{21} \sum x_1\, x_{2.13}$$

$$= \sum x_2\, x_{2.13} - b_{21}(0)$$

$\because \sum x_1\, x_{2.13} = 0$

$$= \sum x_2\, x_{2.13} = R.H.S.$$

Correlation Co-efficient:

$$r_{xy} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}}$$

$$= \frac{\sum (x-\bar{x})(y-\bar{y}) / n}{\sqrt{\frac{\sum (x-\bar{x})^2}{n} \frac{\sum (y-\bar{y})^2}{n}}} \longrightarrow (1)$$

As we know that

Variance $= \sigma^2 = E(X-\mu)$

and $E(X) = \frac{\sum X}{n} = \bar{X}$

So

$$\frac{\sum(x-\bar{x})^2}{n} = \frac{\sum(x^2+\bar{x}^2-2x\bar{x})}{n}$$

$$= \frac{\sum x^2 + \bar{x}^2\sum(1) - 2\bar{x}\sum x}{n}$$

$$= \frac{\sum x^2}{n} + \frac{\bar{x}^2 n}{n} - 2\bar{x}\frac{\sum x}{n}$$

$$= \frac{\sum x^2}{n} + \bar{x}^2 - 2\bar{x}^2$$

$$= \frac{\sum x^2}{n} - \bar{x}^2.$$

$$= \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$= E(X^2) - (E(X))^2$$

$$= V(X) = \sigma_x^2$$

Thus,

$$\sigma_x^2 = \frac{\sum (x-\bar{x})^2}{n} \quad \text{——— (2)}$$

and

Similarly $$\sigma_y^2 = \frac{\sum (y-\bar{y})^2}{n} \quad \text{——— (3)}$$

Thus, equation (1) becomes

$$r_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y}$$

Q. Show that mean of two regression co-effi is the correlation co-efficient.

OR.

Show that $r_{xy} = \sqrt{b_{xy} \cdot b_{yx}}$

Proof:

As

$$r_{xy} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}} \longrightarrow (1)$$

Consider

$$\sum (x-\bar{x})(y-\bar{y}) = \sum (xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})$$

$$= \sum xy - \bar{y}\sum x - \bar{x}\sum y + \bar{x}\bar{y}\sum(1)$$

$$= \sum xy - \bar{y}\sum x - \bar{x}\sum y + \bar{x}\bar{y}(n)$$

$$= \sum xy - n\bar{y}\frac{\sum x}{n} - n\bar{x}\frac{\sum y}{n} + n\bar{x}\bar{y}$$

$$= \sum xy - n\bar{x}\bar{y} - \cancel{n\bar{x}\bar{y}} + \cancel{n\bar{x}\bar{y}}$$

$$= \sum xy - n\frac{\sum x}{n}\frac{\sum y}{n}$$

$$\Rightarrow \sum (x-\bar{x})(y-\bar{y}) = \sum xy - \frac{\sum x \sum y}{n} \rightarrow (2)$$

and

$$\sum (x-\bar{x})^2 = \sum (x^2 + \bar{x}^2 - 2x\bar{x})$$

$$= \sum x^2 + n\bar{x}^2 - 2\bar{x}\sum x$$

$$= \sum x^2 + n\bar{x}^2 - 2\bar{x}n\frac{\sum x}{n}$$

$$= \sum x^2 + n\bar{x}^2 - 2\bar{x}n\bar{x}$$

$$= \sum x^2 + n\bar{x}^2 - 2n\bar{x}^2$$

$$= \sum x^2 - n\bar{x}^2$$

$$= \sum x^2 - n\left(\frac{\sum x}{n}\right)^2$$

$$\Rightarrow \sum (x-\bar{x})^2 = \sum x^2 - \frac{(\sum x)^2}{n} \longrightarrow (3)$$

Similarly,

$$\sum (y-\bar{y})^2 = \sum y^2 - \frac{(\sum y)^2}{n} \longrightarrow (4)$$

Subsituting (2), (3) and (4) in (1), we get

$$r_{xy} = \frac{\sum xy - \sum x \sum y / n}{\sqrt{\left\{\sum x^2 - \frac{(\sum x)^2}{n}\right\}\left\{\sum y^2 - \frac{(\sum y)^2}{n}\right\}}}$$

$$= \frac{n \sum xy - \sum x \sum y}{\sqrt{\left\{n \sum x^2 - (\sum x)^2\right\}\left\{n \sum y^2 - (\sum y)^2\right\}}} \longrightarrow (5)$$

Also, we know that

$$b_{xy} = \frac{n \sum xy - \sum x \sum y}{n \sum y^2 - (\sum y)^2}$$

and

$$b_{yx} = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

$$\Rightarrow b_{xy} \cdot b_{yx} = \frac{(n \sum xy - \sum x \sum y)^2}{\left\{n \sum x^2 - (\sum x)^2\right\}\left\{n \sum y^2 - (\sum y)^2\right\}}$$

Taking square root, we get

$$\sqrt{b_{xy} \cdot b_{yx}} = \frac{\sum xy - \sum x \sum y}{\sqrt{\{n\sum x^2 - (\sum x)^2\}\{n\sum y^2 - (\sum y)^2\}}} \longrightarrow (6)$$

From (5) and (6), we have

$$r_{xy} = \sqrt{b_{xy} \cdot b_{yx}}$$

Remarks:-

(1) Geometric Mean of two Simple regression Co-efficients is the Correlation Co-efficient.

(2) If a Constant is added to each value of variable. It is called Change of origin.

(3) If each value of a variable is multiplied by a Constant, it is called change of scale.

Q, A/2015, A/2014, A/2013

The Correlation Co-efficient is independent of origin and scale.

Solution.

As

$$r_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}} \longrightarrow (1)$$

Let u and v be two new variables defined by

$$u = \frac{x-a}{h} \quad \text{and} \quad v = \frac{y-b}{k}$$

So that $x = a+hu$ and $y = b+kv$

where 'a' and 'b' are new origins and 'h' and 'k' are units of the measurement.

Let r_{xy} denote the correlation co-efficient b/w x and y and r_{uv}, the correlation co-efficient b/w u and v.

Thus equation ① becomes

$$r_{xy} = \frac{\sum\{(a+hu)-(a+h\bar{u})\}\{(b+kv)-(b+k\bar{v})\}}{\sqrt{\sum\{(a+hu)-(a+h\bar{u})\}^2 \sum\{(b+kv)-(b+k\bar{v})\}^2}}$$

$$= \frac{\sum(hu-h\bar{u})(kv-k\bar{v})}{\sqrt{\sum(hu-h\bar{u})^2 \sum(kv-k\bar{v})^2}}$$

$$= \frac{\cancel{hk}\sum(u-\bar{u})(v-\bar{v})}{\cancel{hk}\sqrt{\sum(u-\bar{u})^2 \sum(v-\bar{v})^2}}$$

$$= \frac{\sum (u-\bar{u})(v-\bar{v})}{\sqrt{\sum (u-\bar{u})^2 \sum (v-\bar{v})^2}}$$

$\Rightarrow r_{xy} = r_{uv}$

Hence proved.

Multiple Regression in terms of linear correlation

The sample regression equation of x_1 on x_2 and x_3 is given by

$$x_1 = b_{12.3}\, x_2 + b_{13.2}\, x_3 \quad \text{——} \; (1)$$

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The two normal equations of (1) are given by

$$\sum x_1 x_2 = b_{12.3} \sum x_2^2 + b_{13.2} \sum x_2 x_3$$

and

$$\sum x_1 x_3 = b_{12.3} \sum x_2 x_3 + b_{13.2} \sum x_3^2 \quad \text{——} \; (2)$$

As we know that

$$r_{12} = \frac{\sum (x_1-\bar{x}_1)(x_2-\bar{x}_2)}{\sqrt{\sum (x_1-\bar{x}_1)^2 \sum (x_2-\bar{x}_2)^2}}$$

Since $\bar{x}_1 = \bar{x}_2 = 0$

so

$$r_{12} = \frac{\sum x_1 x_2}{\sqrt{\sum x_1^2 \sum x_2^2}}$$

$$\Rightarrow \sum x_1 x_2 = r_{12} \sqrt{\sum x_1^2 \sum x_2^2}$$

$$= n r_{12} \sqrt{\frac{\sum x_1^2}{n} \cdot \frac{\sum x_2^2}{n}}$$

$$= n r_{12} \sqrt{\frac{\sum x_1^2}{n}} \cdot \sqrt{\frac{\sum x_2^2}{n}}$$

$\cancel{= n r_{12} S_1 S_2}$

Let S_i^2 be the variance of x_i and let r_{ij} be the correlation Co-efficient b/w x_i and x_j.

Then

$$\sum x_1 x_2 = n r_{12} S_1 S_2$$

Similarly,

$$\sum x_1 x_3 = n r_{13} S_1 S_3$$

and

$$\sum x_2 x_3 = n r_{23} S_2 S_3$$

Also

$$\sum x_2^2 = n \frac{\sum x_2^2}{n} = n S_2^2$$

and $\sum x_3^2 = n S_3^2$

Subsituting all these values in equation ②, we get

$$n r_{12} S_1 S_2 = b_{12\cdot3}\, n S_2^2 + b_{13\cdot2}\, n r_{23} S_2 S_3$$
$$n r_{13} S_1 S_3 = b_{12\cdot3}\, n r_{23} S_2 S_3 + b_{13\cdot2}\, n S_3^2$$

$$\Rightarrow r_{12} S_1 = b_{12\cdot3} S_2 + b_{13\cdot2} r_{23} S_3$$
$$r_{13} S_1 = b_{12\cdot3} r_{23} S_2 + b_{13\cdot2} S_3$$

$$\Rightarrow b_{12\cdot3} S_2 + b_{13\cdot2} r_{23} S_3 - r_{12} S_1 = 0$$
$$b_{12\cdot3} S_2 r_{23} + b_{13\cdot2} S_3 - r_{13} S_1 = 0$$

Solving these equations simultaneously for b's, we get

$$\frac{b_{12\cdot3}}{-r_{23} S_3 r_{13} S_1 + r_{12} S_1 S_3} = \frac{-b_{13\cdot2}}{-S_2 r_{13} S_1 + r_{12} S_1 r_{23} S_2}$$

$$= \frac{1}{S_2 S_3 - r_{23} S_3 r_{23} S_3}$$

$$\Rightarrow b_{12\cdot3} = \frac{r_{12} S_1 S_3 - r_{13} r_{23} S_1 S_3}{S_2 S_3 - r_{23}^2 S_3^2}$$

$$= \frac{S_1 \cancel{S_3}\,(r_{12} - r_{13} r_{23})}{S_2 \cancel{S_3}\,(1 - r_{23}^2)}$$

$$\Rightarrow b_{12.3} = \frac{s_1(r_{12} - r_{13}r_{23})}{s_2(1 - r_{23}^2)}$$

and

$$b_{13.2} = \frac{s_1 \cancel{s_2}(r_{13} - r_{12}r_{23})}{\cancel{s_2} s_3(1 - r_{23}^2)}$$

$$\Rightarrow b_{13.2} = \frac{s_1(r_{13} - r_{12}r_{23})}{s_3(1 - r_{23}^2)}$$

Now equation ① becomes

$$x_1 = \frac{s_1(r_{12} - r_{13}r_{23})x_2}{s_2(1 - r_{23}^2)} + \frac{s_1(r_{13} - r_{12}r_{23})x_3}{s_3(1 - r_{23}^2)}$$

$$\Rightarrow \frac{x_1}{s_1} = \left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)\frac{x_2}{s_2} + \left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right)\frac{x_3}{s_3}$$

which is the multiple regression of x_1 on x_2 and x_3 in terms of standard deviations and liner correlation coefficients of the valiable involved.

The regression equations of x_2 on x_1 and x_3 and of x_3 on x_1 and x_2 are given by

$$x_2 = b_{21.3}\, x_1 + b_{23.1}\, x_3$$

$$x_3 = b_{31.2}\, x_1 + b_{32.1}\, x_2$$

we can easily find that

$$b_{21.3} = \frac{s_2 (r_{21} - r_{23} r_{13})}{s_1 (1 - r_{13}^2)}$$

$\because r_{12} = r_{21}$
$r_{13} = r_{31}$
$r_{23} = r_{32}$

$$b_{23.1} = \frac{s_2 (r_{23} - r_{21} r_{13})}{s_3 (1 - r_{13}^2)}$$ A/2017

and

$$b_{31.2} = \frac{s_3 (r_{31} - r_{32} r_{12})}{s_1 (1 - r_{12}^2)}$$

$$b_{32.1} = \frac{s_3 (r_{32} - r_{31} r_{12})}{s_2 (1 - r_{12}^2)}$$

Thus,

$$x_2 = \frac{s_2 (r_{21} - r_{23} r_{13})}{s_1 (1 - r_{13}^2)} x_1 + \frac{s_2 (r_{23} - r_{21} r_{13})}{s_3 (1 - r_{13}^2)} x_3$$

and

$$x_3 = \frac{s_3 (r_{31} - r_{32} r_{12})}{s_1 (1 - r_{12}^2)} x_1 + \frac{s_3 (r_{32} - r_{31} r_{12})}{s_2 (1 - r_{12}^2)} x_2$$

$$\Rightarrow \frac{x_2}{s_2} = \left(\frac{r_{12} - r_{23} r_{13}}{1 - r_{13}^2} \right) \frac{x_1}{s_1} + \left(\frac{r_{23} - r_{12} r_{13}}{1 - r_{13}^2} \right) \frac{x_3}{s_3}$$

and

$$\frac{x_3}{s_3} = \left(\frac{r_{13} - r_{23} r_{12}}{1 - r_{12}^2}\right)\frac{x_1}{s_1} + \left(\frac{r_{23} - r_{13} r_{12}}{(1 - r_{12}^2)}\right)\frac{x_2}{s_2}$$

Prove that A/2018 Multiple Correlation co-efficient

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2 r_{12} r_{23} r_{13}}{1 - r_{23}^2}}$$

Proof.

Let us denote the estimated value of x_1 by $\hat{x}_1$. Then by definition, we have

$$R_{1.23} = \frac{Cov(x_1, \hat{x}_1)}{\sqrt{Var(x_1)\, Var(\hat{x}_1)}}$$

where $\hat{x}_1 = b_{12.3} x_2 + b_{13.2} x_3$

$$\Rightarrow R_{1.23} = \frac{\sum x_1 \hat{x}_1}{\sqrt{\sum x_1^2 \sum (\hat{x}_1)^2}} \quad (1)$$

Now,

$$\sum x_1 \hat{x}_1 = \sum x_1 (b_{12.3} x_2 + b_{13.2} x_3)$$

$$= b_{12.3} \sum x_1 x_2 + b_{13.2} \sum x_1 x_3 \rightarrow (2)$$

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Since $b_{12.3} = \dfrac{S_1(r_{12} - r_{13}r_{23})}{S_2(1 - r_{23}^2)}$

and

$$b_{13.2} = \frac{S_1(r_{13} - r_{12}r_{23})}{S_3(1 - r_{23}^2)}$$

$$\sum x_1 x_2 = n r_{12} S_1 S_2$$

and

$$\sum x_1 x_3 = n r_{13} S_1 S_3$$

Therefore, equation ② becomes

$$\sum x_1 \hat{x}_1 = \frac{S_1(r_{12} - r_{13}r_{23})}{\cancel{S_2}(1 - r_{23}^2)} n r_{12} S_1 \cancel{S_2} + \frac{S_1(r_{13} - r_{12}r_{23})}{\cancel{S_3}(1 - r_{23}^2)} n r_{13} S_1 \cancel{S_3}$$

$$= \frac{nS_1^2(r_{12} - r_{13}r_{23})r_{12}}{1 - r_{23}^2} + \frac{nS_1^2(r_{13} - r_{12}r_{23})r_{13}}{1 - r_{23}^2}$$

$$= \frac{nS_1^2}{1 - r_{23}^2}\left[(r_{12} - r_{13}r_{23})r_{12} + (r_{13} - r_{12}r_{23})r_{13}\right]$$

$$= \frac{nS_1^2}{1 - r_{23}^2}\left[r_{12}^2 - r_{12}r_{13}r_{23} + r_{13}^2 - r_{12}r_{13}r_{23}\right]$$

$$= \frac{n S_1^2}{1-r_{23}^2}\left(r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} r_{23}\right) \text{ —— } (3)$$

$$\sum x_1^2 = n \frac{\sum x_1^2}{n} = n S_1^2 \text{ —— } (4)$$

and

$$\sum (\hat{x}_1)^2 = \sum \left(b_{12.3}\, x_2 + b_{13.2}\, x_3 \right)^2$$

$$= \sum \left(b_{12.3}^2\, x_2^2 + b_{13.2}^2\, x_3^2 + 2\, b_{12.3}\, b_{13.2}\, x_2 x_3 \right)$$

$$= \sum b_{12.3}^2\, x_2^2 + \sum b_{13.2}^2\, x_3^2 + \sum 2\, b_{12.3}\, b_{13.2}\, x_2 x_3$$

$$= b_{12.3}^2 \sum x_2^2 + b_{13.2}^2 \sum x_3^2 + 2\, b_{12.3}\, b_{13.2} \sum x_2 x_3$$

$$= \frac{S_1^2 (r_{12} - r_{13} r_{23})^2}{\cancel{S_2^2} (1-r_{23}^2)^2}\, n \cancel{S_2^2} + \frac{S_1^2 (r_{13} - r_{12} r_{23})^2}{\cancel{S_3^2} (1-r_{23}^2)^2}\, n \cancel{S_3^2}$$

$$+ 2\, \frac{S_1 (r_{12} - r_{13} r_{23})}{\cancel{S_2} (1-r_{23}^2)} \times \frac{S_1 (r_{13} - r_{12} r_{23})}{\cancel{S_3} (1-r_{23}^2)}\, n\, r_{23} \cancel{S_2} \cancel{S_3}$$

$$= \frac{nS_1^2 (r_{12} - r_{13} r_{23})^2}{(1-r_{23}^2)^2} + \frac{nS_1^2 (r_{13} - r_{12} r_{23})^2}{(1-r_{23}^2)^2}$$

$$+ 2nS_1^2 \frac{(r_{12} - r_{13} r_{23})(r_{13} - r_{12} r_{23})}{(1-r_{23}^2)^2} r_{23}$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2} \left[(r_{12} - r_{13} r_{23})^2 + (r_{13} - r_{12} r_{23})^2 + 2(r_{12} - r_{13} r_{23})(r_{13} - r_{12} r_{23}) r_{23} \right]$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2} \left[\begin{array}{l} r_{12}^2 + r_{13}^2 r_{23}^2 - 2 r_{12} r_{13} r_{23} + \\ r_{13}^2 + r_{12}^2 r_{23}^2 - 2 r_{12} r_{13} r_{23} + \\ 2(r_{12} r_{13} r_{23} - r_{12}^2 r_{23}^2 - r_{13}^2 r_{23}^2 \\ \quad + r_{12} r_{13} r_{23}^3) \end{array} \right]$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2} \left[\begin{array}{l} r_{12}^2 + r_{13}^2 r_{23}^2 \cancel{- 2 r_{12} r_{13} r_{23}} \\ r_{13}^2 + r_{12}^2 r_{23}^2 - 2 r_{12} r_{13} r_{23} \\ \cancel{+ 2 r_{12} r_{13} r_{23}} - 2 r_{12}^2 r_{23}^2 - \\ 2 r_{13}^2 r_{23}^2 + 2 r_{12} r_{13} r_{23}^3 \end{array} \right]$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2} \left[\begin{array}{l} r_{12}^2 + r_{13}^2 - r_{13}^2 r_{23}^2 \\ - r_{12}^2 r_{23}^2 - 2 r_{12} r_{13} r_{23} \\ + 2 r_{12} r_{13} r_{23}^3 \end{array} \right]$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2}\left[r_{12}^2 + r_{13}^2 - r_{13}^2 r_{23}^2 - r_{12}^2 r_{23}^2 - 2r_{12}r_{13}r_{23}(1-r_{23}^2) \right]$$

$$= \frac{nS_1^2}{(1-r_{23}^2)^2}\left[r_{12}^2(1-r_{23}^2) + r_{13}^2(1-r_{23}^2) - 2r_{12}r_{13}r_{23}(1-r_{23}^2) \right]$$

$$= \frac{nS_1^2}{1-r_{23}^2}\left(r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23} \right) \quad \text{—— (5)}$$

subsituting (3), (4) and (5) in (1), we get

$$R_{1\cdot 23} = \frac{\frac{nS_1^2}{1-r_{23}^2}\left(r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}\right)}{\sqrt{nS_1^2 \times \frac{nS_1^2}{1-r_{23}^2}\left(r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}\right)}}$$

$$= \frac{\cancel{nS_1^2}/\cancel{1-r_{23}^2}\,\left(r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}\right)}{\cancel{nS_1^2}/\cancel{\sqrt{1-r_{23}^2}}\,\sqrt{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}}$$

$$= \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1-r_{23}^2}}$$

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Prove that $R_{2.31} = \sqrt{\dfrac{r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}}{1 - r_{13}^2}}$

Solution,

Let us denote the estimated value of x_2 by $\hat{x}_2$. Then by definition, we have

$$R_{2.31} = \frac{Cov(x_2, \hat{x}_2)}{\sqrt{Var(x_2)\, Var(\hat{x}_2)}}$$

where $\hat{x}_2 = b_{21.3}\, x_1 + b_{23.1}\, x_3$

$$\Rightarrow R_{2.31} = \frac{\sum x_2 \hat{x}_2}{\sqrt{\sum x_2^2 \sum (\hat{x}_2)^2}} \longrightarrow (1)$$

Now,

$$\sum x_2 \hat{x}_2 = \sum x_2 (b_{21.3}\, x_1 + b_{23.1}\, x_3)$$

$$= b_{21.3} \sum x_1 x_2 + b_{23.1} \sum x_2 x_3 \longrightarrow (2)$$

Since

$$b_{21.3} = \frac{S_2 (r_{21} - r_{23} r_{13})}{S_1 (1 - r_{13}^2)}$$

$$b_{23.1} = \frac{S_2 (r_{23} - r_{21} r_{13})}{S_3 (1 - r_{13}^2)}$$

and $\sum x_1 x_2 = n r_{12} S_1 S_2$

$\sum x_2 x_3 = n r_{23} S_2 S_3$

Therefore, equation (2) implies that

$$\sum x_2 \hat{x}_2 = \frac{S_2 (r_{21} - r_{23} r_{13})}{\cancel{S_1}(1 - r_{13}^2)} n r_{12} \cancel{S_1} S_2$$

$$+ \frac{S_2 (r_{23} - r_{12} r_{13})}{\cancel{S_3}(1 - r_{13}^2)} n r_{23} S_2 \cancel{S_3}$$

$$= \frac{n S_2^2}{1 - r_{13}^2} \left(r_{12}^2 - r_{12} r_{13} r_{23} + r_{23}^2 - r_{12} r_{13} r_{23} \right)$$

$$= \frac{n S_2^2}{1 - r_{13}^2} \left(r_{23}^2 + r_{21}^2 - 2 r_{12} r_{23} r_{13} \right) \quad \text{——— (3)}$$

and

$$\sum x_2^2 = n \frac{\sum x_2^2}{n} = n S_2^2 \quad \text{——— (4)}$$

$$\sum (\hat{x}_2)^2 = \sum (b_{21.3} x_1 + b_{23.1} x_3)^2$$

$$= b_{21.3}^2 \sum x_1^2 + b_{23.1}^2 \sum x_3^2$$

$$+ 2 b_{21.3} b_{23.1} \sum x_1 x_3$$

$$= \frac{S_2^2 (r_{21} - r_{23} r_{13})^2}{\cancel{S_1^2}(1-r_{13}^2)^2}\, n \cancel{S_1^2} + \frac{S_2^2 (r_{23}-r_{21}r_{13})^2}{\cancel{S_3^2}(1-r_{13}^2)^2}\, n \cancel{S_3^2}$$

$$+ 2\,\frac{S_2(r_{21}-r_{23}r_{13})}{\cancel{S_1}(1-r_{13}^2)} \times \frac{S_2(r_{23}-r_{21}r_{13})}{\cancel{S_3}(1-r_{13}^2)}\, n\, r_{13} \cancel{S_1}\cancel{S_3}$$

$$= \frac{nS_2^2}{(1-r_{13}^2)^2}\left[\begin{aligned} & r_{21}^2 + r_{23}^2 r_{13}^2 - 2 r_{21}r_{23}r_{13} \\ & + r_{23}^2 + r_{21}^2 r_{13}^2 - 2r_{23}r_{21}r_{13} \\ & + 2(r_{21}r_{23}r_{13} - r_{21}^2 r_{13}^2 - r_{23}^2 r_{13}^2 \\ & \quad + r_{23}r_{21}r_{13}^3) \end{aligned}\right]$$

$$= \frac{nS_2^2}{(1-r_{13}^2)^2}\left[\begin{aligned} & r_{21}^2 + r_{21}^2 r_{13}^2 + r_{23}^2 + r_{23}^2 r_{13}^2 \\ & - 4 r_{12} r_{13} r_{23} + 2 r_{12} r_{13} r_{23} \\ & - 2 r_{21}^2 r_{13}^2 - 2 r_{23}^2 r_{13}^2 \\ & + 2 r_{21} r_{23} r_{13}^3 \end{aligned}\right]$$

$$= \frac{nS_2^2}{(1-r_{13}^2)^2}\left[\begin{aligned} & r_{21}^2 - r_{21}^2 r_{13}^2 + r_{23}^2 - r_{23}^2 r_{13}^2 \\ & - 2 r_{12} r_{13} r_{23} + 2 r_{12} r_{23} r_{13}^2 \end{aligned}\right]$$

$$= \frac{nS_2^2}{(1-r_{13}^2)^2}\left[\begin{aligned} & r_{21}^2(1-r_{13}^2) + r_{23}^2(1-r_{13}^2) \\ & - 2 r_{12} r_{13} r_{23}(1-r_{13}^2) \end{aligned}\right]$$

$$= \frac{nS_2^2}{1-r_{13}^2}\left(r_{21}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}\right) \quad \text{—} (5)$$

Subsituting (3), (4) and (5) in (1), we get

$$R_{2.31} = \frac{\frac{nS_2^2}{1-r_{13}^2}\left(r_{21}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}\right)}{\sqrt{nS_2^2 \times \frac{nS_2^2}{1-r_{13}^2}\left(r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}\right)}}$$

$$= \frac{\cancel{nS_2^2}\left(\frac{r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}}{1-r_{13}^2}\right)}{\cancel{nS_2^2}\sqrt{\frac{\left(r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}\right)}{1-r_{13}^2}}}$$

$$\Rightarrow R_{2.31} = \sqrt{\frac{r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{13}}{1-r_{13}^2}}$$ as req

Similarly, we can show

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{12}r_{13}r_{23}}{1-r_{12}^2}}$$

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31/5/2019

Coefficient of Partial Correlation

The partial Correlation Co-efficient between x_1 and x_2 when the influence of x_3 has been eliminated, is given by

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$$r_{12.3} = \frac{\sum x_{1.3}\, x_{2.3}}{\sqrt{\sum x_{1.3}^2 \sum x_{2.3}^2}} \quad \text{——— (1)}$$

where

$$x_{1.3} = x_1 - b_{13} x_3$$

and $x_{2.3} = x_2 - b_{23} x_3$

Since $b_{13} = \dfrac{n \sum x_1 x_3 - \sum x_1 \sum x_3}{n \sum (x_3)^2 - (\sum x_3)^2}$

and

$$\sum (x_1 - \bar{x}_1)(x_3 - \bar{x}_3) = \frac{n \sum x_1 x_3 - \sum x_1 \sum x_3}{n}$$

$$\sum (x_3 - \bar{x}_3)^2 = \frac{n \sum x_3^2 - (\sum x_3)^2}{n}$$

so

$$b_{13} = \frac{n \sum (x_1 - \bar{x}_1)(x_3 - \bar{x}_3)}{n \sum (x_3 - \bar{x}_3)^2}$$

$$= \frac{\dfrac{\sum (x_1 - \bar{x}_1)(x_3 - \bar{x}_3)}{n}}{\dfrac{\sum (x_3 - \bar{x}_3)}{n}}$$

$$= \frac{Cov(x_1, x_3)}{S_3^2}$$

$$= \frac{Cov(x_1, x_3)}{S_3^2} \times \frac{S_1}{S_1}$$

$$= \frac{Cov(x_1, x_3)}{S_1 S_3} \cdot \frac{S_1}{S_3}$$

$$= r_{13} \frac{S_1}{S_3}$$

$$\Rightarrow \boxed{b_{13} = r_{13} \frac{S_1}{S_3}}$$

Similarly,

$$b_{23} = r_{23} \frac{S_2}{S_3}$$

Thus, $x_{1.3} = x_1 - r_{13} \frac{S_1}{S_3} x_3$

and $x_{2.3} = x_2 - r_{23} \frac{S_2}{S_3} x_3$

Now,

$$\sum x_{1.3}\, x_{2.3} = \sum \left(x_1 - r_{13}\frac{s_1}{s_3}x_3\right)\left(x_2 - r_{23}\frac{s_2}{s_3}x_3\right)$$

$$= \sum \left(x_1x_2 - r_{23}\frac{s_2}{s_3}x_1x_3 - r_{13}\frac{s_1}{s_3}x_2x_3 + r_{13}r_{23}\frac{s_1}{s_3}x_3 \cdot \frac{s_2}{s_3}x_3\right)$$

$$= \sum x_1x_2 - r_{23}\frac{s_2}{s_3}\sum x_1x_3 - r_{13}\frac{s_1}{s_3}\sum x_2x_3 + r_{13}r_{23}\frac{s_1s_2}{s_3^2}\sum x_3^2 .$$

$$= n r_{12}s_1s_2 - r_{23}\frac{s_2}{\cancel{s_3}}\cdot n r_{13}s_1\cancel{s_3} - r_{13}\frac{s_1}{\cancel{s_3}}n r_{23}s_2\cancel{s_3} + r_{13}r_{23}\frac{s_1s_2}{\cancel{s_3^2}}n\cancel{s_3^2}$$

$$= n\left[r_{12}s_1s_2 - r_{13}r_{23}s_1s_2 \cancel{- r_{13}r_{23}s_1s_2} \cancel{+ r_{13}r_{23}s_1s_2}\right]$$

$$= n(r_{12}s_1s_2 - r_{13}r_{23}s_1s_2)$$

$$= n s_1s_2(r_{12} - r_{13}r_{23}) \quad \text{——} \quad 2$$

$$\sum x_{1.3}^2 = \sum \left(x_1 - r_{13} \frac{s_1}{s_3} x_3\right)^2$$

$$= \sum x_1^2 + r_{13}^2 \frac{s_1^2}{s_3^2} \sum x_3^2 - 2 r_{13} \frac{s_1}{s_3} \sum x_1 x_3$$

$$= n s_1^2 + r_{13}^2 \frac{s_1^2}{\cancel{s_3^2}} n \cancel{s_3^2} - 2 r_{13} \frac{s_1}{\cancel{s_3}} n r_{13} s_1 \cancel{s_3}$$

$$= n s_1^2 + n r_{13}^2 s_1^2 - 2 r_{13}^2 n s_1^2$$

$$= n s_1^2 (1 + r_{13}^2 - 2 r_{13}^2)$$

$$= n s_1^2 (1 - r_{13}^2) \quad \text{———} \quad (3)$$

and

$$\sum x_{2.3}^2 = \sum \left(x_2 - r_{23} \frac{s_2}{s_3} x_3\right)^2$$

$$= \sum x_2^2 + r_{23}^2 \frac{s_2^2}{s_3^2} \sum x_3^2 - 2 r_{23} \frac{s_2}{s_3} \sum x_2 x_3$$

$$= n s_2^2 + r_{23}^2 \frac{s_2^2}{\cancel{s_3^2}} n \cancel{s_3^2} - 2 r_{23} \frac{s_2}{\cancel{s_3}} n r_{23} s_2 \cancel{s_3}$$

$$= n s_2^2 + n r_{23}^2 s_2^2 - n r_{23}^2 s_2^2$$

$$= n s_2^2 (1 - r_{23}^2) \quad \text{———} \quad (4)$$

Subsituting ②, ③ and ④ in ①,
we get

$$r_{12.3} = \frac{nS_1S_2\,(r_{12} - r_{13}r_{23})}{\sqrt{nS_1^2(1-r_{13}^2)\cdot nS_2^2(1-r_{23}^2)}}$$

$$= \frac{n\cancel{S_1}\cancel{S_2}\,(r_{12} - r_{13}r_{23})}{\cancel{n}\cancel{S_1}S_2\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$\Rightarrow r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

Similarly, we can find

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

and

$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}$$

Alternate

The partial correlation co-efficient between x_1 and x_2 when the influence of x_3 has been eliminated, is defined as the geometric mean of the regression co-efficients $b_{12.3}$ and $b_{21.3}$ of the two partial regression lines of x_1 on x_2 and of x_2 on x_1 respectively

i.e.

$$r_{12.3} = \sqrt{b_{12.3} \times b_{21.3}} \quad \text{—— (1)}$$

Since

$$b_{12.3} = \frac{s_1 (r_{12} - r_{13} r_{23})}{s_2 (1 - r_{23}^2)}$$

and

$$b_{21.3} = \frac{s_2 (r_{21} - r_{23} r_{13})}{s_1 (1 - r_{13}^2)}$$

Therefore,

$$r_{12.3} = \sqrt{\frac{\cancel{s_1}(r_{12} - r_{13} r_{23})}{\cancel{s_2}(1 - r_{23}^2)} \times \frac{\cancel{s_2}(r_{12} - r_{13} r_{23})}{\cancel{s_1}(1 - r_{13}^2)}}$$

$$= \frac{r_{12} - r_{13} r_{23}}{\sqrt{(1 - r_{23}^2)(1 - r_{13}^2)}}$$

Similarly other two

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Write the six partial regression Co-efficients and show that the partial Correlation Co-efficients can be expressed as a geometric mean of some two of the above six partial regression Co-efficients. Justify your answer.

Solution:

Partial regression Coefficients.

$b_{12.3} = ?$ $b_{13.2} = ?$

$b_{21.3} = ?$ $b_{23.1} = ?$ already

$b_{31.2} = ?$ $b_{32.1} = ?$ Calculated.

Already done,

$$r_{xy} = \sqrt{b_{xy} \times b_{yx}}$$

Justification:

$$r_{12.3} = \sqrt{b_{12.3} \times b_{21.3}}$$

given on last page.

Regression For Continuous Variable

Bi-Variate Regression / Regression Equation:

If $f(x,y)$ is the joint probability density function of variable X & Y The the regression equation of Y on X is denoted by $\mu_{Y|x}$ and is defined as

Conditional Prob.

$$\mu_{Y|x} = E(Y|x) = \int_{-\infty}^{\infty} y \cdot f(Y|x)\,dy$$

Similarly the Regression Equ. of X on Y

$$\mu_{X|y} = E(X|y) = \int_{-\infty}^{\infty} x \cdot f(x|y)\,dx$$

where

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

and

$$g(x) = \begin{cases} \int_{-\infty}^{\infty} f(x,y)\,dy & , -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

$$h(y) = \begin{cases} \int_{-\infty}^{\infty} f(x,y)\,dx & -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Example:

Given the two random variables X and Y that have the joint density

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$$f(x,y) = \begin{cases} x \cdot e^{-x(1+y)} & : x>0, y>0 \\ 0 & : \text{elsewhere} \end{cases}$$

Find the regression eq of Y on X and X on Y and sketch the regression curve.

Soln:

The regression equation of Y on X is

$$\mu_{Y|x} = E(Y|x) = \int_{-\infty}^{\infty} y f(y|x) dy \quad \cdots\cdots \rightarrow (1)$$

$$f(y|x) = \frac{f(x,y)}{g(x)} \quad \cdots\cdots \rightarrow (1)$$

Now.

$$g(x) = \int_0^{\infty} x e^{-x(1+y)} dy = x e^{-x} \int_0^{\infty} e^{-xy} dy$$

$$g(x) = x e^{-x} \left| \frac{e^{-xy}}{-x} \right|_0^{\infty} = -e^{-x}(0-1) = e^{-x}$$

Putting in (1)

$$f(y|x) = \frac{x e^{-x(1+y)}}{e^{-x}} = \frac{x e^{-x} \cdot e^{-xy}}{e^{-x}} = x e^{-xy}$$

$f(y|x) = x e^{-xy}$ for $y>0$

$f(y|x) = 0$, elsewhere

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$$\mu_{y|x} = \int_{-\infty}^{\infty} y f(y|x)\,dy$$

$$= \int_{-\infty}^{0} y f(y|x)\,dy + \int_{0}^{\infty} y f(y|x)\,dy$$

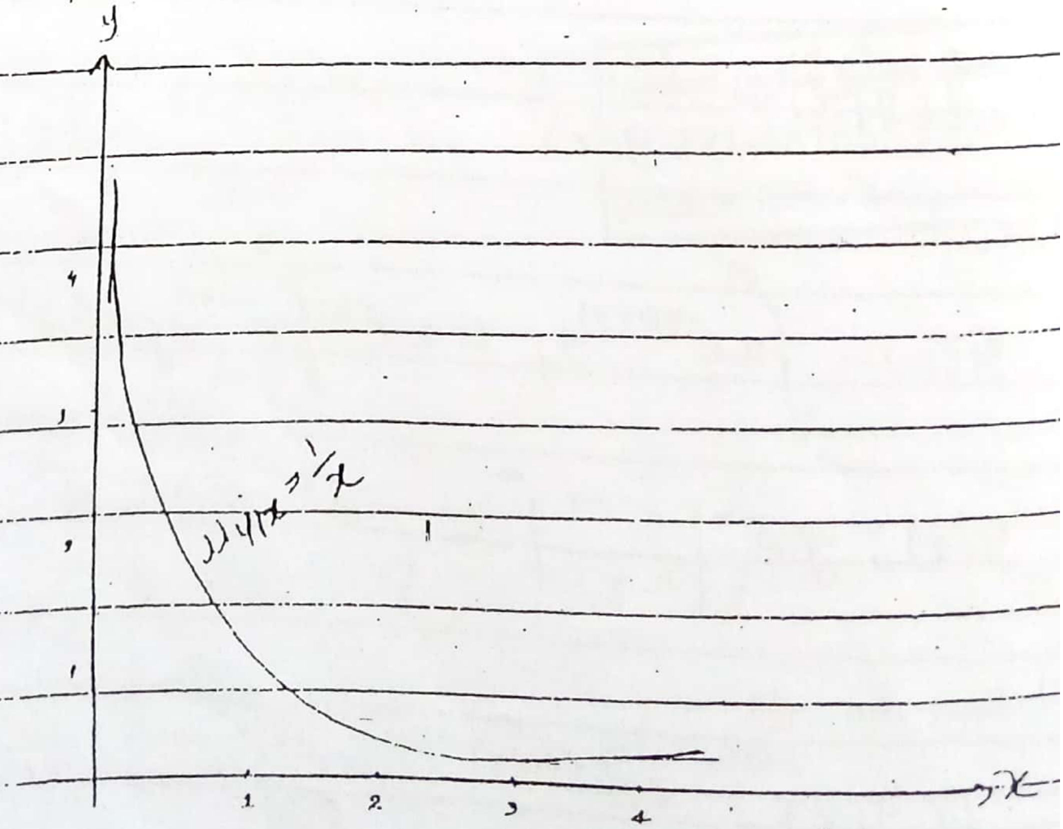
$$= 0 + \int_{0}^{\infty} y \cdot x e^{-xy}\,dy$$

$$= \int_{0}^{\infty} x \cdot y e^{-xy}\,dy = x \int_{0}^{\infty} y e^{-xy}\,dy$$

$$= x\left[y \cdot \frac{e^{-xy}}{-x} \Big|_0^{\infty} - \int_0^{\infty} -\frac{e^{-xy}}{x} \cdot 1\,dy \right]$$

$$= x\left(0 + \frac{1}{x} \cdot \frac{e^{-xy}}{-x} \Big|_0^{\infty} \right)$$

$$\mu_{y|x} = x\left(\frac{-1}{x^2}\right)\left(e^{-\infty} - e^{-0}\right) = \frac{-1}{x}(0-1) = \frac{1}{x}$$



FUNCTIONS OF RANDOM VARIABLES

STaTiSTics

1 Introduction

In this chapter we shall concern ourselves with the problem of finding the probability distributions or densities of **functions of one or more random variables**. That is, given a set of random variables $X_1, X_2, \ldots, X_n$ and their joint probability distribution or density, we shall be interested in finding the probability distribution or density of some random variable $Y = u(X_1, X_2, \ldots, X_n)$. This means that the values of Y are related to those of the X's by means of the equation

$$y = u(x_1, x_2, \ldots, x_n).$$

Several methods are available for solving this kind of problem. The ones we shall discuss in the next four sections are called the **distribution function technique**, the **transformation technique**, and the **moment-generating function technique**. Although all three methods can be used in some situations, in most problems one technique will be preferable (easier to use than the others). This is true, for example, in some instances where the function in question is linear in the random variables $X_1, X_2, \ldots, X_n$, and the moment-generating function technique yields the simplest derivations.

2 Distribution Function Technique

A straightforward method of obtaining the probability density of a function of continuous random variables consists of first finding its distribution function and then its probability density by differentiation. Thus, if $X_1, X_2, \ldots, X_n$ are continuous random variables with a given joint probability density, the probability density of $Y = u(X_1, X_2, \ldots, X_n)$ is obtained by first determining an expression for the probability

$$F(y) = P(Y \leqq y) = P[u(X_1, X_2, \ldots, X_n) \leqq y]$$

and then differentiating to get

$$f(y) = \frac{dF(y)}{dy}$$

From Chapter 7 of *John E. Freund's Mathematical Statistics with Applications*, Eighth Edition. Irwin Miller, Marylees Miller.

Example 3:

$$F(y) = \int_0^y \int_0^{y-x_2} 6e^{-3x_1-2x_2}\, dx_1\, dx_2$$

$$= 6\int_0^y \frac{e^{-3x_1-2x_2}}{-3}\Big|_0^{y-x_2} dx_2$$

$$= -2\int_0^y \left(e^{-3y+3x_2-2x_2} - e^{-2x_2}\right) dx_2$$

$$= -2\int_0^y \left(e^{-3y+x_2} - e^{-2x_2}\right) dx_2$$

$$= -2\int_0^y e^{-3y+x_2}\, dx_2 + 2\int_0^y e^{-2x_2}\, dx_2$$

$$= -2\left[e^{-3y+x_2}\right]_0^y + 2\left[\frac{e^{-2x_2}}{-2}\right]_0^y$$

$$= -2\left(e^{-2y} - e^{-3y}\right)$$

$$-\left(e^{-2y} - e^{0}\right)$$

$$= -2e^{-2y} + 2e^{-3y}$$

$$- e^{-2y} + 1$$

$$F(y) = 1 + 2e^{-3y} - 3e^{-2y}$$

EXAMPLE 1

If the probability density of X is given by

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = X^3$.

Solution

Letting $G(y)$ denote the value of the distribution function of Y at y, we can write

$$G(y) = P(Y \leqq y)$$

$$= P(X^3 \leqq y)$$

$$= P(X \leqq y^{1/3})$$

$$= \int_0^{y^{1/3}} 6x(1-x)\, dx$$

$$G(y) = 3y^{2/3} - 2y$$

and hence $g(y) = \frac{d}{dy} G(y) = 3 \cdot \frac{2}{3} y^{\frac{2}{3}-1} - 2$

$$\Rightarrow g(y) = 2(y^{-1/3} - 1)$$

for $0 < y < 1$; elsewhere, $g(y) = 0$. In Exercise 15 the reader will be asked to verify this result by a different technique.

EXAMPLE 2

If $Y = |X|$, show that

$$g(y) = \begin{cases} f(y) + f(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $f(x)$ is the value of the probability density of X at x and $g(y)$ is the value of the probability density of Y at y. Also, use this result to find the probability density of $Y = |X|$ when X has the standard normal distribution.

Solution

For $y > 0$ we have

$$G(y) = P(Y \leqq y)$$

$$= P(|X| \leqq y)$$

$$= P(-y \leqq X \leqq y)$$

$$= F(y) - F(-y)$$

$$P(-y \le X \le y) = \int_{-y}^{y} f(x)\, dx = F(x)\Big|_{-y}^{y} = G(y)$$

and, upon differentiation, $g(y) = \frac{d}{dy} G(y)$

$$g(y) = f(y) + f(-y)$$

Exercises

1. If X has an exponential distribution with the parameter θ, use the distribution function technique to find the probability density of the random variable $Y = \ln X$.

2. If the probability density of X is given by

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and $Y = X^2$, find
(a) the distribution function of Y;
(b) the probability density of Y.

(2) (a) $F(y) = P(Y \le y)$

$= P(X^2 \le y) = P(X \le y^{1/2})$

$= \int_0^{y^{1/2}} 2x e^{-x^2}\, dx$

$= -e^{-x^2} \Big|_0^{y^{1/2}} = -e^{-y} + 1$

(b) $f(y) = \frac{d}{dy} F(y)$

$= \frac{d}{dy}(1 - e^{-y})$

$= +e^{-y}$

$\Rightarrow f(y) = \begin{cases} e^{-y} & ; 0 < y < 1 \\ 0 & ; \text{otherwise} \end{cases}$

3. If X has the uniform density with the parameters $\alpha = 0$ and $\beta = 1$, use the distribution function technique to find the probability density of the random variable $Y = \sqrt{X}$.

4. If the joint probability density of X and Y is given by

see last page

$$f(x, y) = \begin{cases} 4xye^{-(x^2+y^2)} & \text{for } x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and $Z = \sqrt{X^2 + Y^2}$, find
(a) the distribution function of Z;
(b) the probability density of Z.

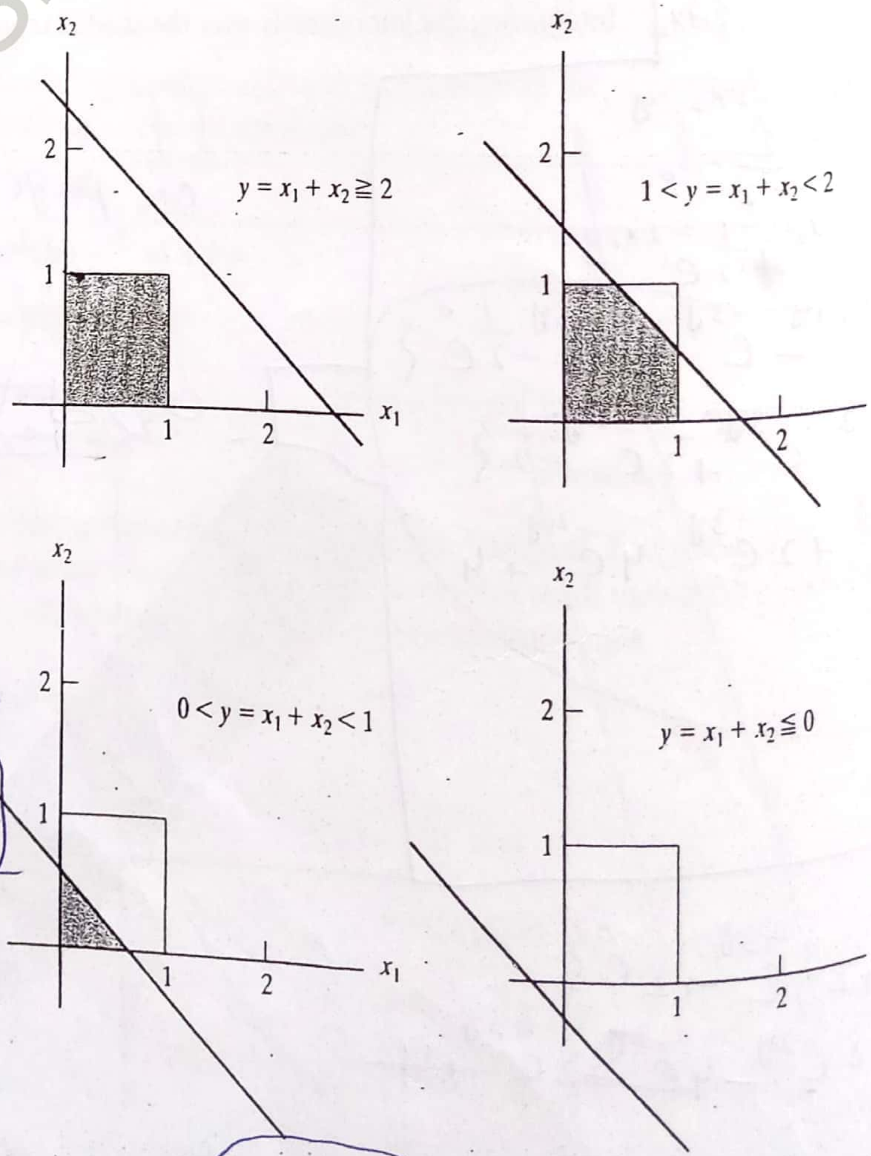


Figure 2. Diagram for Exercise 6.

If we ... distribution with $n = 4$ and $\theta = \frac{1}{2}$,

$p = 1/2$, $q = 1/2$

$f(x) = \binom{n}{x} p^x q^{n-x}$

$= \binom{n}{x} p^x q^{-x} q^n$

$= \binom{n}{x} \left(\frac{p}{q}\right)^x q^n$

$= \binom{n}{x} \left(\frac{1/2}{1/2}\right)^x q^n$

$= \binom{n}{x} \left(\frac{1}{2}\right)^n = \binom{4}{x}\left(\frac{1}{2}\right)^4$, $q = 1/2$

$$f(x) = \binom{4}{x}\left(\frac{1}{2}\right)^4 \quad \text{for } x = 0, 1, 2, 3, 4$$

getting

$$g(y) = f\left(\frac{1}{y} - 1\right) = \binom{4}{\frac{1}{y} - 1}\left(\frac{1}{2}\right)^4 \quad \text{for } y = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

Note that in the preceding example the probabilities remained unchanged; the only difference is that in the result they are associated with the various values of Y instead of the corresponding values of X. That is all there is to the **transformation** (or **change-of-variable**) **technique** in the discrete case as long as the relationship is one-to-one. If the relationship is not one-to-one, we may proceed as in the following example.

EXAMPLE 5

With reference to Example 4, find the probability distribution of the random variable $Z = (X - 2)^2$.

Solution

Calculating the probabilities $h(z)$ associated with the various values of Z, we get

$$h(0) = f(2) = \frac{6}{16}$$

$$h(1) = f(1) + f(3) = \frac{4}{16} + \frac{4}{16} = \frac{8}{16}$$

$$h(4) = f(0) + f(4) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$$

and hence

z	0	1	4
$h(z)$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

$\Sigma h(z) = 1$

To perform a transformation of variable in the continuous case, we shall assume that the function given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, so the inverse function, given by $x = w(y)$, exists for all the corresponding values of y and is differentiable except where $u'(x) = 0$.† Under these conditions, we can prove the following theorem.

EXAMPLE 8

If $F(x)$ is the value of the distribution function of the continuous random variable X at x, find the probability density of $Y = F(X)$.

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ding
y

Solution

As can be seen from Figure 6, the value of Y corresponding to any particular value of X is given by the area under the curve, that is, the area under the graph of the density of X to the left of x. Differentiating $y = F(x)$ with respect to x, we get

$$\frac{dy}{dx} = F'(x) = f(x)$$

and hence

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f(x)}$$

provided $f(x) \neq 0$. It follows from Theorem 1 that

$$g(y) = f(x) \cdot \left| \frac{1}{f(x)} \right| = 1$$

for $0 < y < 1$, and we can say that y has the uniform density with $\alpha = 0$ and $\beta = 1$.

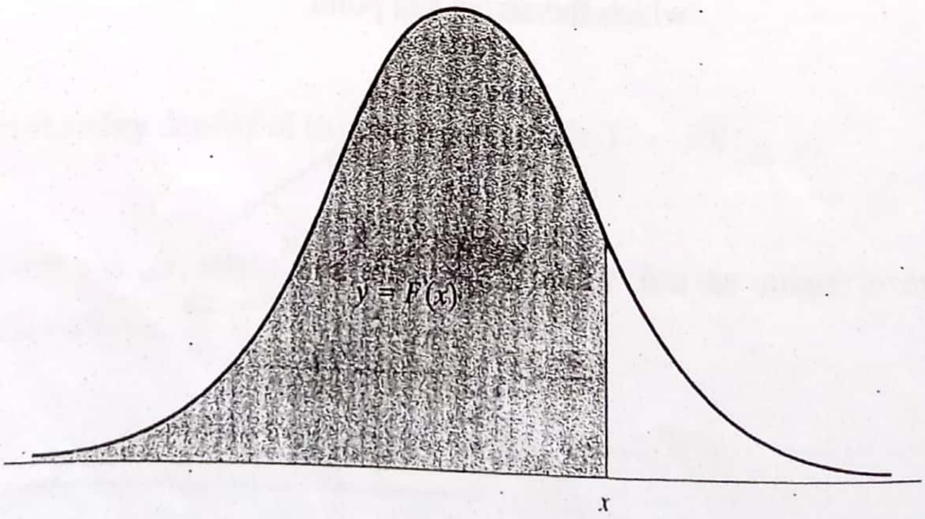

Figure 6. Diagram for Example 8.

EXAMPLE 10

If X_1 and X_2 are independent random variables having Poisson distributions with the parameters λ_1 and λ_2, find the probability distribution of the random variable $Y = X_1 + X_2$.

Solution

Since X_1 and X_2 are independent, their joint distribution is given by

$$f(x_1, x_2) = \frac{e^{-\lambda_1}(\lambda_1)^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}(\lambda_2)^{x_2}}{x_2!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1)^{x_1}(\lambda_2)^{x_2}}{x_1!x_2!}$$

for $x_1 = 0, 1, 2, \ldots$ and $x_2 = 0, 1, 2, \ldots$. Since $y = x_1 + x_2$ and hence $x_1 = y - x_2$, we can substitute $y - x_2$ for x_1, getting

$$g(y, x_2) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!}$$

for $y = 0, 1, 2, \ldots$ and $x_2 = 0, 1, \ldots, y$, for the joint distribution of Y and X_2. Then, summing on x_2 from 0 to y, we get

$$h(y) = \sum_{x_2=0}^{y} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!}$$

× and ÷ by $y!$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \cdot \sum_{x_2=0}^{y} \frac{y!}{x_2!(y-x_2)!}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}$$

after factoring out $e^{-(\lambda_1+\lambda_2)}$ and multiplying and dividing by $y!$. Identifying the summation at which we arrived as the binomial expansion of $(\lambda_1+\lambda_2)^y$, we finally get

$$h(y) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^y}{y!} \quad \text{for } y = 0, 1, 2, \ldots$$

and we have thus shown that the sum of two independent random variables having Poisson distributions with the parameters λ_1 and λ_2 has a Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2$.

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EXAMPLE 11

If the joint probability density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = \dfrac{X_1}{X_1+X_2}$.

$$y = \frac{x_1}{x_1+x_2}$$
$$\Rightarrow x_1 + x_2 = \frac{x_1}{y}$$
$$\Rightarrow x_2 = \frac{x_1}{y} - x_1$$
$$\Rightarrow x_2 = x_1\left(\frac{1}{y} - 1\right)$$
$$\Rightarrow x_2 = \frac{x_1(1-y)}{y}$$

Solution

Since y decreases when x_2 increases and x_1 is held constant, we can use Theorem 1 to find the joint density of X_1 and Y. Since $y = \dfrac{x_1}{x_1+x_2}$ yields $x_2 = x_1 \cdot \dfrac{1-y}{y}$ and hence

$$\frac{\partial x_2}{\partial y} = -\frac{x_1}{y^2}$$

$$x_2 = x_1\left(\frac{1}{y} - 1\right)$$
$$\Rightarrow \frac{\partial x_2}{\partial y} = x_1\left(-\frac{1}{y^2}\right)$$

it follows that

$$g(x_1, y) = f(x_1, x_2)\left|\frac{\partial x_2}{\partial y}\right|$$

$$g(x_1, y) = e^{-x_1/y}\left|-\frac{x_1}{y^2}\right| = \frac{x_1}{y^2} \cdot e^{-x_1/y}$$

for $x_1 > 0$ and $0 < y < 1$. Finally, integrating out x_1 ~~and changing the variable of integration to $u = x_1/y$~~, we get

$$h(y) = \int_0^\infty \frac{x_1}{y^2} \cdot e^{-x_1/y}\, dx_1$$

Put $u = x_1/y$

$du = dx_1/y$

$x_1 \to 0 \Rightarrow u \to 0$

$x_1 \to \infty \Rightarrow u \to \infty$

$$= \int_0^\infty \frac{x_1}{y} e^{-\frac{x_1}{y}} \frac{dx_1}{y}$$

$$= \int_0^\infty u \cdot e^{-u}\, du$$

$$= \int_0^\infty u^{2-1} e^{-u}\, du$$

$$= \Gamma(2)$$

$$= 1!$$

$$= 1$$

for $0 < y < 1$, and $h(y) = 0$ elsewhere. Thus, the random variable Y has the uniform density with $\alpha = 0$ and $\beta = 1$. (Note that in Exercise 7 the reader was asked to show this by the distribution function technique.)

THEOREM 2. Let $f(x_1,x_2)$ be the value of the joint probability density of the continuous random variables X_1 and X_2 at (x_1,x_2). If the functions given by $y_1 = u_1(x_1,x_2)$ and $y_2 = u_2(x_1,x_2)$ are partially differentiable with respect to both x_1 and x_2 and represent a one-to-one transformation for all values within the range of X_1 and X_2 for which $f(x_1,x_2) \neq 0$, then, for these values of x_1 and x_2, the equations $y_1 = u_1(x_1,x_2)$ and $y_2 = u_2(x_1,x_2)$ can be uniquely solved for x_1 and x_2 to give $x_1 = w_1(y_1,y_2)$ and $x_2 = w_2(y_1,y_2)$, and for the corresponding values of y_1 and y_2, the joint probability density of $Y_1 = u_1(X_1,X_2)$ and $Y_2 = u_2(X_1,X_2)$ is given by

$$g(y_1,y_2) = f[w_1(y_1,y_2), w_2(y_1,y_2)] \cdot |J|$$

Here, J, called the **Jacobian** of the transformation, is the determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Elsewhere, $g(y_1,y_2) = 0$.

We shall not prove this theorem, but information about Jacobians and their applications can be found in most textbooks on advanced calculus. There they are used mainly in connection with multiple integrals, say, when we want to change from rectangular coordinates to polar coordinates or from rectangular coordinates to spherical coordinates.

EXAMPLE 12

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With reference to the random variables X_1 and X_2 of Example 11, find

(a) the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1 + X_2}$;

(b) the marginal density of Y_2.

Solution

(a) Solving $y_1 = x_1 + x_2$ and $y_2 = \frac{x_1}{x_1 + x_2}$ for x_1 and x_2, we get $x_1 = y_1 y_2$ and $x_2 = y_1(1 - y_2)$, and it follows that

$\because y_1 = x_1 + x_2$
$\Rightarrow x_2 = y_1 - x_1$
$\Rightarrow x_2 = y_1 - y_1 y_2$
$\Rightarrow x_2 = y_1(1 - y_2)$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$\Rightarrow J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix}$$

$= -y_1 y_2 - y_1(1 - y_2)$
$= -\cancel{y_1 y_2} - y_1 + \cancel{y_1 y_2} = -y_1$

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Since the transformation is one-to-one, mapping the region $x_1 > 0$ and $x_2 > 0$ in the x_1x_2-plane into the region $y_1 > 0$ and $0 < y_2 < 1$ in the y_1y_2-plane, we can use Theorem 2 and it follows that

$$g(y_1, y_2) = e^{-y_1}|-y_1| = y_1e^{-y_1}$$

for $y_1 > 0$ and $0 < y_2 < 1$; elsewhere, $g(y_1, y_2) = 0$.

(b) Using the joint density obtained in part (a) and integrating out y_1, we get

$$\begin{aligned} h(y_2) &= \int_0^{\infty} g(y_1, y_2)\,dy_1 \\ &= \int_0^{\infty} y_1e^{-y_1}\,dy_1 = \int_0^{\infty} y_1^{2-1} e^{-y_1}\,dy_1 \\ &= \Gamma(2) \\ &= 1 \end{aligned}$$

$$\Rightarrow h(y_2) = \begin{cases} 1 & ; 0<y_2<1 \\ 0 & ; \text{elsewhere} \end{cases}$$

for $0 < y_2 < 1$; elsewhere, $h(y_2) = 0$.

EXAMPLE 13

If the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y = X_1 + X_2$ and $Z = X_2$;

(b) the marginal density of Y.

Note that in Exercise 6 the reader was asked to work the same problem by the distribution function technique.

Solution

(a) Solving $y = x_1 + x_2$ and $z = x_2$ for x_1 and x_2, we get $x_1 = y - z$ and $x_2 = z$, so that

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} \Rightarrow \quad J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Because this transformation is one-to-one, mapping the region $0 < x_1 < 1$ and $0 < x_2 < 1$ in the x_1x_2-plane into the region $z < y < z+1$ and $0 < z < 1$ in the yz-plane (see Figure 7), we can use Theorem 2 and we get

$$g(y, z) = 1 \cdot |1| = 1$$

for $z < y < z+1$ and $0 < z < 1$; elsewhere, $g(y, z) = 0$.

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(b)

$$h(y_1) = \int_0^{y_1}\int_0^{y_1-y_3} e^{-y_1}\, dy_2\, dy_3$$

$$= \int_0^{y_1} y_2 e^{-y_1} \Big|_0^{y_1-y_3} dy_3$$

$$= \int_0^{y_1} (y_1-y_3) e^{-y_1}\, dy_3$$

$$= \int_0^{y_1} y_1 e^{-y_1}\, dy_3 - \int_0^{y_1} y_3 e^{-y_1}\, dy_3$$

$$= y_1 y_3 e^{-y_1}\Big|_0^{y_1} - \frac{y_3^2}{2} e^{-y_1}\Big|_0^{y_1}$$

$$= y_1^2 e^{-y_1} - \frac{y_1^2}{2} e^{-y_1}$$

$$= \frac{1}{2} y_1^2 e^{-y_1}$$

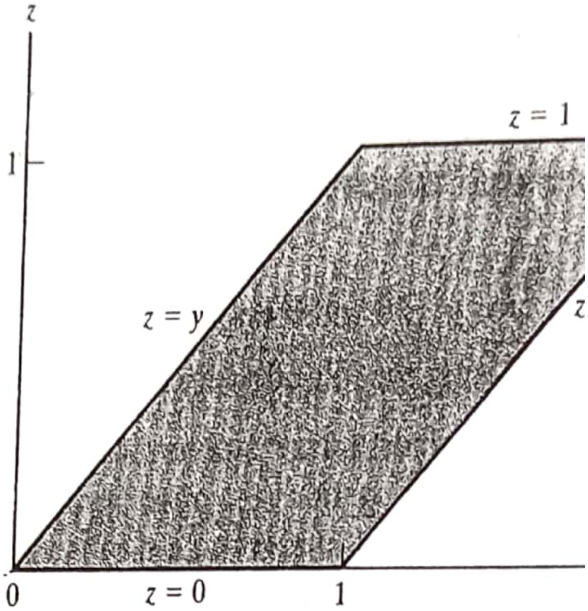
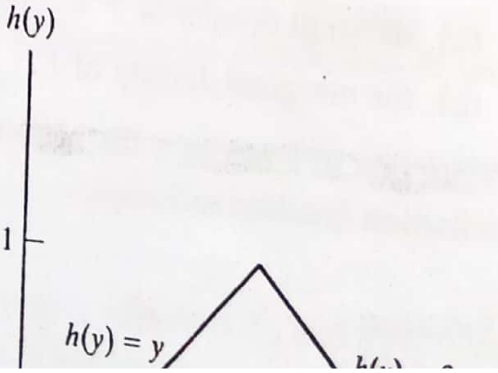


Figure 7. Transformed sample space for

(b) Integrating out z separately for $y \leqq 0$, $0 < y < 1$, $1 <$

$$h(y) = \begin{cases} 0 & \text{for} \\ \int_0^y 1 \cdot dz = y & \text{for} \\ \int_{y-1}^1 1 \cdot dz = 2 - y & \text{for} \\ 0 & \text{for} \end{cases}$$

and to make the density function continuous, we l
shown that the sum of the given random variables h
ity density whose graph is shown in Figure 8.



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EXAMPLE 14

A/2018 If the joint probability density of X_1, X_2, and X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & \text{for } x_1 > 0, x_2 > 0, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_2$, and $Y_3 = X_3$;

(b) the marginal density of Y_1.

Solution

(a) Solving the system of equations $y_1 = x_1 + x_2 + x_3$, $y_2 = x_2$, and $y_3 = x_3$ for x_1, x_2, and x_3, we get $x_1 = y_1 - y_2 - y_3$, $x_2 = y_2$, and $x_3 = y_3$. It follows that

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} \Rightarrow$$

$$J = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and, since the transformation is one-to-one, that

$$g(y_1, y_2, y_3) = e^{-y_1} \cdot |1|$$
$$= e^{-y_1}$$

for $y_2 > 0$, $y_3 > 0$, and $y_1 > y_2 + y_3$; elsewhere, $g(y_1, y_2, y_3) = 0$.

(b) Integrating out y_2 and y_3, we get

$$h(y_1) = \int_0^{y_1} \int_0^{y_1 - y_3} e^{-y_1}\, dy_2\, dy_3$$

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$$= \frac{1}{2} y_1^2 \cdot e^{-y_1}$$

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for $y_1 > 0$; $h(y_1) = 0$ elsewhere. Observe that we have shown that the sum of three independent random variables having the gamma distribution with $\alpha = 1$ and $\beta = 1$ is a random variable having the gamma distribution with $\alpha = 3$ and $\beta = 1$.